Data-driven stability analysis methods

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Notable approaches for stability analysis of nonlinear systems

- Model-based PWA Lyapunov functions offering tractability through piecewise system approximations [Julian, Guivant, and Desages, 1999];
- Neural Lyapunov approaches to learn stability certificates from data [Kim and Kim, 2024; Kolter and Manek, 2019];
- Interval and polynomial-based techniques as alternative methods [Le Mézo, Jaulin, and Zerr, 2017; Martin and Allgöwer, 2024];
- Data-driven methods constructing Lyapunov functions by solving an optimisation problem [Tacchi, Lian, and Jones, 2025].

Outline

1. **Data-based framework**

2. **PWA Lyapunov-based RoA identification**

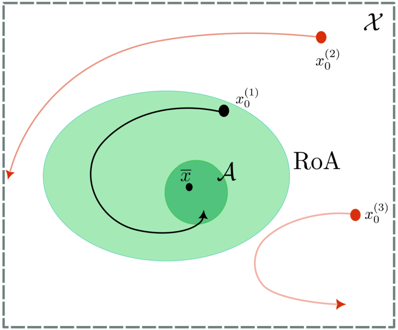
3. **Outer approximation of the RoA using SoS-programming**

Data-based framework

Objective

We consider the system with unknown, Lipschitz continuous dynamics :

$$\dot{x} = f(x) \tag{1}$$



Dataset

$\Longrightarrow$ Availability of a dataset $\mathcal{D} := \{(x_d, f_d)\}_{1 \leq d \leq N_d}$ with $f_d = f(x_d)$

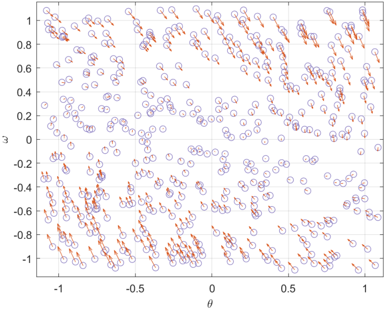
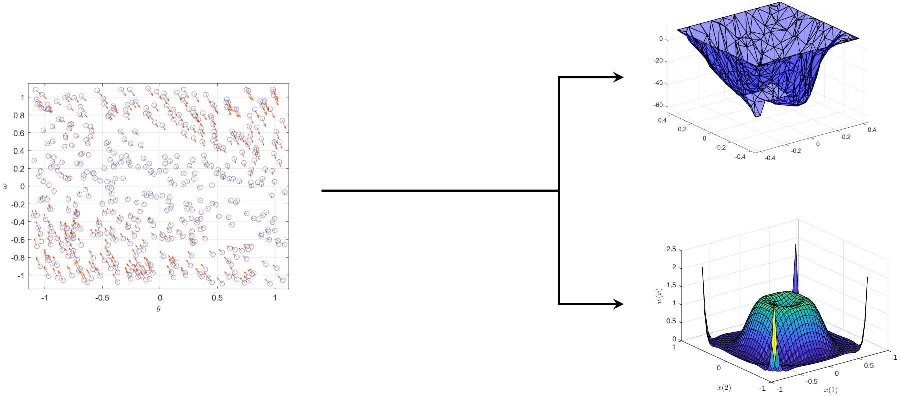


Figure: Example of a fixed dataset

Proposed methods



PWA Lyapunov-based RoA identification

Method 1

Constructing Piece-Wise Affine Lyapunov candidates based on data:

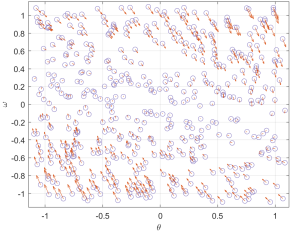


Figure: Dataset

$\longrightarrow$

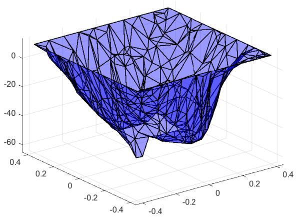


Figure: PWA Lyapunov candidate

Tessellation

The region $\overline{\mathcal{X} \setminus \mathcal{A}}$ will be subdivided into an N_c-piece tessellation $\{Y_c\}_{1\leq c\leq N_c}$:

$$\Longrightarrow \text{Dataset } \mathcal{D} := \{(x_d, f_d)\}_{1\leq d\leq N_d} + \text{tessellation } \{Y_c\}_{1\leq c\leq N_c}$$

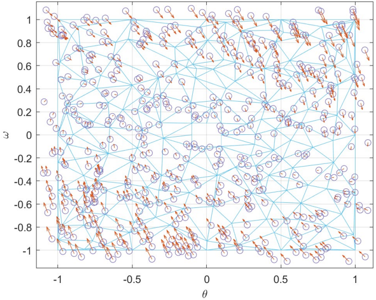


Figure: Dataset and tessellation

Data-driven PWA Lyapunov construction

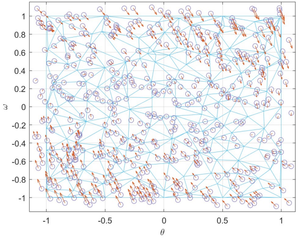


Figure: Dataset and tessellation example

$\Longrightarrow$ A piece-wise affine Lyapunov function is defined as such:

$$\forall c \in \{1, ..., N_c\},\ \forall x \in Y_c \subset \overline{\mathcal{X} \setminus \mathcal{A}},$$

$$V(x) = V_c(x) = {g_c}^\top x + b_c \tag{2a}$$

$$x \in \mathbf{L}_\alpha^V \subset \mathcal{X} \Rightarrow \dot{V}(x) = \nabla V^\top f(x) < 0 \tag{2b}$$

with $\mathbf{L}_\alpha^V := \{x \in \mathcal{X} | V(x) < \alpha\}$

Levelset

The existence of a Lyapunov candidate as defined in (2) proves the local asymptotic stability of $\bar{x}$ in $\mathbf{L}_\alpha^V$ according to the following:

Theorem

Based on the assumptions and the Krasovskii-LaSalle theorem, if there exists a level set $\mathbf{L}_\alpha^V := \{x \in \mathcal{X} | V(x) \leq \alpha\}$ *such that* $x \in \mathbf{L}_\alpha^V \subset \mathcal{X} \Rightarrow \dot{V}(x) = \nabla V^\top f(x) < 0$*, then:*

- $\mathbf{L}_\alpha^V$ *is positively invariant;*
- $\mathbf{L}_\alpha^V$ *is an extended subset of the region of attraction of* $\bar{x}$*.*

Optimisation problem

The following optimisation problem is used to find V:

$$\min_{g_c, b_c, s_c} \sum s_c$$

$$\text{s. t.} \;\; \forall\, c,\; c' \;\in \{1, ..., N_c\},\; v_{i,c} \in Y_c$$

$$-\mu \leq s_c \tag{3}$$

$$v_{i,c} \in Y_{c'} \;\Rightarrow\; V(v_{i,c}) = {g_c}^\top v_{i,c} + b_c = {g_{c'}}^\top v_{i,c} + b_{c'} \tag{4}$$

$$\nabla V^\top f(v_{i,c}) = g_c \top f(v_{i,c}) \leq s_c \tag{5}$$

with N_c the number of cells, $v_{i,c}$ a vertex in the tessellation, s_c an introduced slack variable, and $\mu \in]0, \infty)$.

Lipschitz continuity

By defining $\sum_{d=1}^{N_d} \gamma_{d,c} = g_c$, condition $\nabla V^\top f(x) = g_c{}^\top f(x) \leq s_{i,c}$ for $x \in Y_c$ can become:

$$\sum_{d=1}^{N_d} \gamma_{d,c}{}^\top f_d + ||\gamma_{d,c}|| M ||v_{i,c} - x_d|| \leq s_{i,c}$$

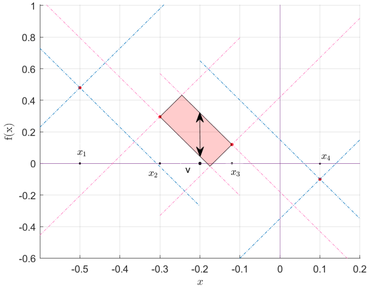
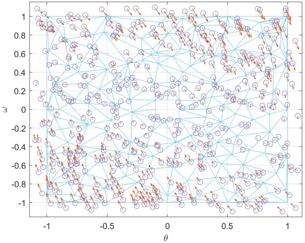


Figure: Illustrating Lipschitz inequality : $||f(x) - f(y)|| \leq M||x - y||, \quad \forall x, y \in \mathcal{X}$

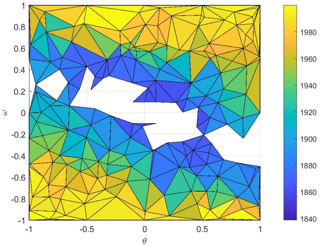
Challenge

Considering the following damped pendulum example:

$$\dot{x} = \begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} \omega \\ -\sin(\theta) - 2\omega \end{pmatrix} \tag{6}$$



(a) Pendulum dataset and tessellation



(b) Resulting Lyapunov candidate

$\Longrightarrow$ **Positive slack variables fail to prove Lyapunov decrease**

Challenge

According to Tacchi et al.[1], the dataset and tessellation should be **refined** enough to provide good coverage of $\overline{\mathcal{X} \setminus \mathcal{A}}$. To tackle:

- no a priori condition on the quality of the refinement;
- boundlessly refining the dataset and tessellation is impractical;
- monotonous certification procedure.

[1] "Robustly learning regions of attraction from fixed data", Tacchi, Lian, and Jones, IEEE Transactions on Automatic Control 2025

Iterative construction of invariant sets

Consider $\mathcal{X} = \mathcal{X}_0 \supset \ldots \supset \mathcal{X}_K$, $\mathcal{A} = \mathcal{A}_0 \supset \ldots \supset \mathcal{A}_K$, $\mathcal{D}_0, \ldots, \mathcal{D}_K$, and $\mathcal{C}_0, \ldots, \mathcal{C}_K$ (the corresponding uncertified cells with non-negative slack):

For an iteration $k \leq k_{max}$:

1. Generate $\{Y_c\}_{1 \leq c \leq N_{c_k}}$ s.t. $\mathcal{X}_k = \cup_{c=1}^{N_{c_k}} Y_c$
2. Generate a sufficiently informative dataset $\mathcal{D}_k$
3. Run optimisation problem; get V_k, $\mathbf{L}_{\alpha_k}^{V_k}$, $\mathcal{C}_k$:
    - if $\mathbf{L}_{\alpha_k}^{V_k} \not\supseteq \mathcal{C}_{k-1}$, iteration fails;
    - else if $\mathcal{C}_k \subseteq \mathcal{A}_0 = \mathcal{A}$, algorithm terminates;
    - else, define $\mathcal{X}_{k+1} \subset \mathbf{L}_{\alpha_k}^{V_k}$, $\mathcal{A}_{k+1} \subset \mathcal{A}_k$ for next iteration.

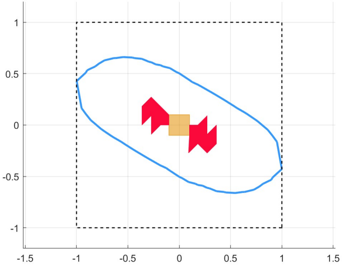


Figure: An iteration k

Iterative construction of invariant sets

Consider $\mathcal{X} = \mathcal{X}_0 \supset \ldots \supset \mathcal{X}_K$, $\mathcal{A} = \mathcal{A}_0 \supset \ldots \supset \mathcal{A}_K$, $\mathcal{D}_0, \ldots, \mathcal{D}_K$, and $\mathcal{C}_0, \ldots, \mathcal{C}_K$ (the corresponding uncertified cells with non-negative slack):

For an iteration $k \leq k_{max}$:

1. Generate $\{Y_c\}_{1 \leq c \leq N_{c_k}}$ s.t. $\mathcal{X}_k = \cup_{c=1}^{N_{c_k}} Y_c$
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3. Run optimisation problem; get V_k, $\mathbf{L}_{\alpha_k}^{V_k}$, $\mathcal{C}_k$:
   - if $\mathbf{L}_{\alpha_k}^{V_k} \not\supseteq \mathcal{C}_{k-1}$, iteration fails;
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   - else, define $\mathcal{X}_{k+1} \subset \mathbf{L}_{\alpha_k}^{V_k}$, $\mathcal{A}_{k+1} \subset \mathcal{A}_k$ for next iteration.

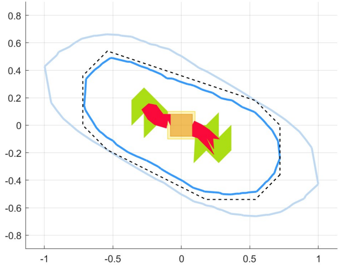


Figure: The next iteration k+1

Numerical simulation

Using YALMIP and MPT3 toolboxes, and MOSEK as a solver, the solution was obtained in **five iterations** for the following initial sets:

Set	**Value**
$\mathcal{X}$	$[-1;1]^2$
$\mathcal{A}$	$[-0.1;0.1]^2$

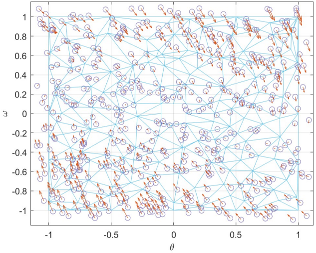
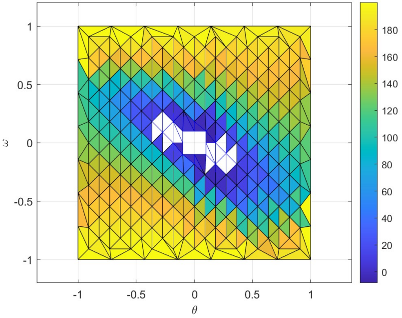


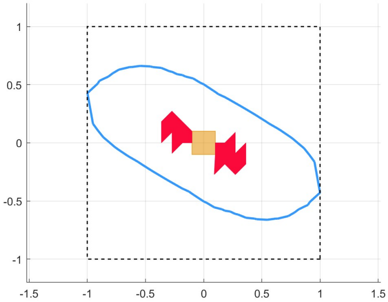
Figure: Initial dataset

N.B.: The following simulation is based on an informed tessellation.

Iteration 1



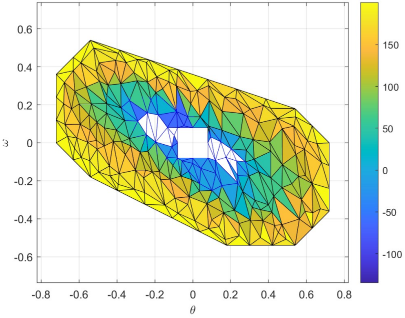
(a) $N_{d_k} = 300$, $N_{v_k} = 252$



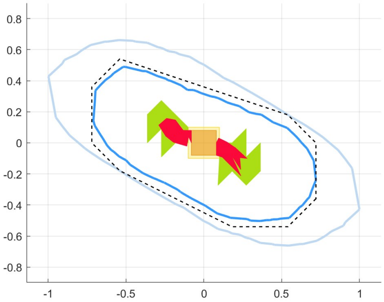
(b) $\alpha_k = 111$

Figure: $k = 1$

Iteration 2



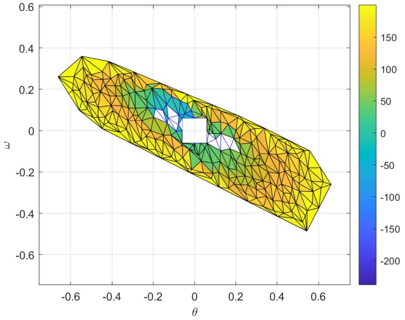
(a) $N_{d_k} = 349$, $N_{v_k} = 221$



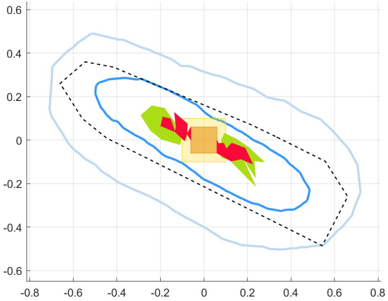
(b) $\alpha_k = 174$

Figure: $k = 2$

Iteration 3



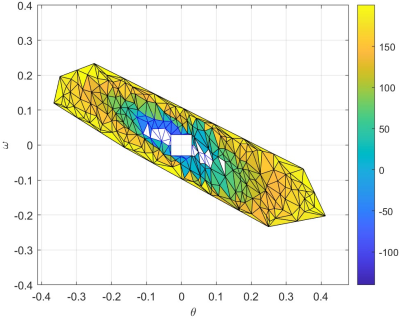
(a) $N_{d_k} = 346$, $N_{v_k} = 238$



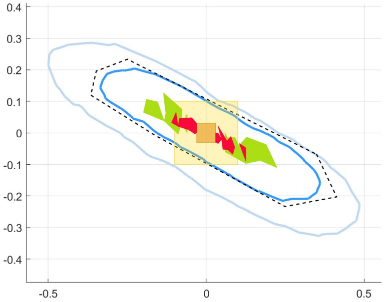
(b) $\alpha_k = 146$

Figure: $k = 3$

Iteration 4



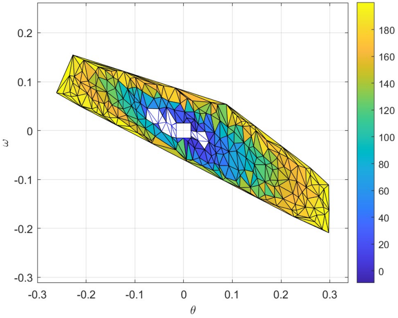
(a) $N_{d_k} = 348$, $N_{v_k} = 241$



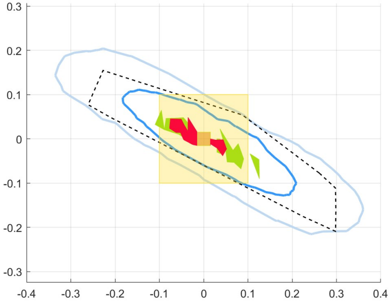
(b) $\alpha_k = 180$

Figure: $k = 4$

Iteration 5



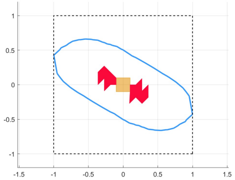
(a) $N_{d_k} = 370$, $N_{v_k} = 239$



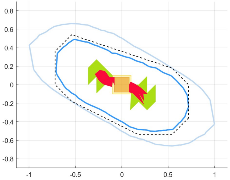
(b) $\alpha_k = 145$

Figure: $k = 5$

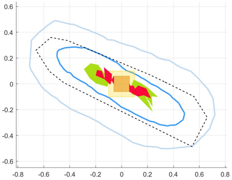
All iterations



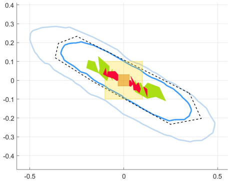
(a) $k = 1$



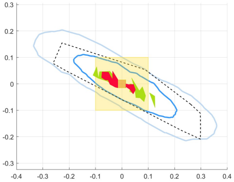
(b) $k = 2$



(c) $k = 3$



(a) $k = 4$



(b) $k = 5$

Obtained invariant sets

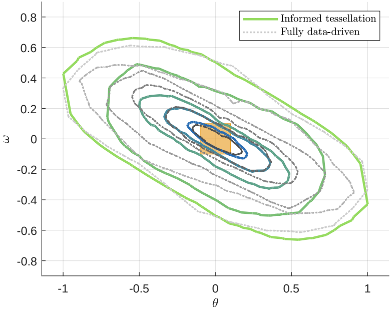


Figure: Resulting level sets

$\Longrightarrow$ The **choice of tessellation** strongly influences the speed of convergence.

Outer approximation of the RoA using SoS-programming

Method 2

Data-driven approximation of the region of attraction using Sum-of-Squares programming:

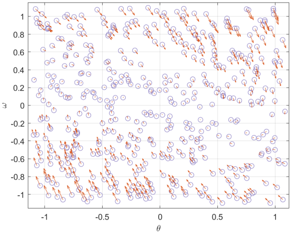


Figure: Dataset

$\longrightarrow$

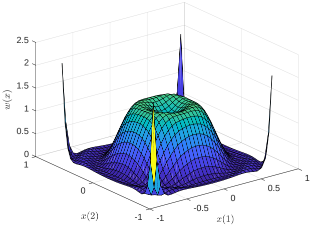


Figure: Polynomial RoA approximation

Approaching the indicator function

The objective is to approach the indicator function the RoA corresponding to the convergence in finite time T of the system.

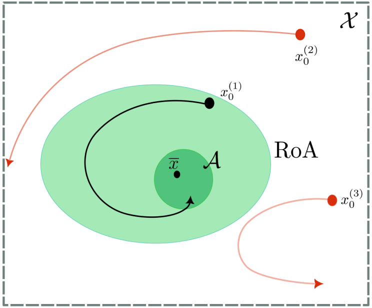


Figure: Region of attraction illustration

Approaching the indicator function

The objective is to approach the indicator function the RoA corresponding to the convergence in finite time T of the system.

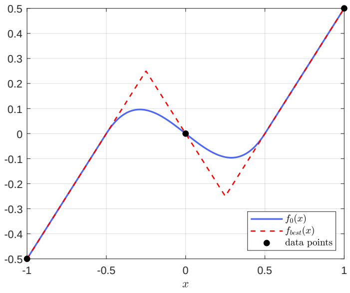


Figure: 1D example function

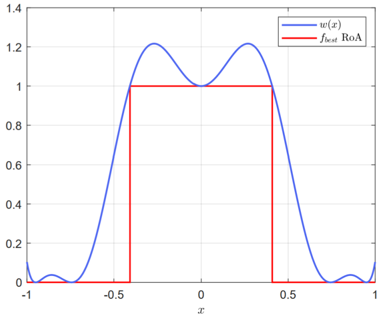


Figure: Approaching the RoA

Optimisation problem

The following optimisation problem is used to find the polynomial w:

$$\underset{\substack{v \in C^1([0,T]\times\mathcal{X}) \\ w \in C^0(\mathcal{X})}}{\text{minimise}} \int_{\mathcal{X}} w(x)\, dx \tag{7a}$$

$$\text{s.t.} \quad \frac{\partial v}{\partial t}(t,x) + y^\top \frac{\partial v}{\partial x}(t,x) \leq 0 \qquad \forall (t,x,y) \in [0,T] \times \Gamma_{\mathcal{D}} \tag{7b}$$

$$v(T,x) \geq 0 \qquad \forall x \in \mathcal{A} \tag{7c}$$

$$w(x) \geq v(0,x) + 1 \qquad \forall x \in \mathcal{X} \tag{7d}$$

$$w(x) \geq 0 \qquad \forall x \in \mathcal{X} \tag{7e}$$

with v a polynomial function working as an intermediary variable, and $\Gamma_{\mathcal{D}}$ the set of tuples (x, y) such that y verifies the Lipschitz continuity of f; y here takes all the possible values of f.

Numerical application

Application on a 2D system:

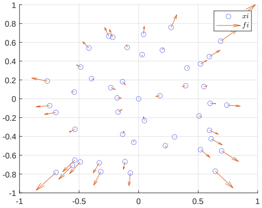


Figure: Dataset

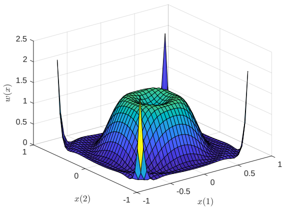


Figure: Obtained polynomial

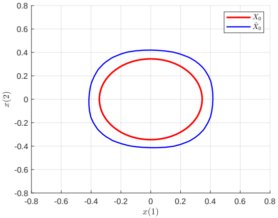


Figure: Outer approximation

Conclusion & outlook

- For method 1:
  - Problem solved while avoiding complexity from refining datasets or tessellations using an iterative method;
  - Switching the euclidean norm to the infinity norm is to be explored.
- For method 2:
  - Promising approach with strong mathematical guarantees;
  - Future works includes adding a state-space splitting to counter the numerical problems.

For code, please visit:

`github.com/OumaymaK`

References

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