Safe Control of Nonlinear Systems using Data-Driven Set-Valued Models

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Safe learning-based control – our setting

- We consider a discrete-time nonlinear system of the form:

$$x_{t+1} = f(x_t, u_t) + g(x_t, u_t), x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m$$

where f is a known map and g is unknown.

- Given a data set

$$D = \{(x_i, u_i, x_i^+) |\, x_i^+ = f(x_i, u_i) + g(x_i, u_i), i = 1, \dots, N\}$$

we aim at synthesizing a controller for our system such that the safety constraints hold:

$$x_t \in \mathbb{X}, u_t \in \mathbb{U}, \forall t \in \mathbb{N}$$

- All results in the presentation can be adapted if the dynamics has bounded disturbances.

Safe learning-based control – taxonomy*

- Learning the model vs. learning the controller
- Probabilistic guarantees vs. robust guarantees
- Parametric approaches vs. nonparametric approaches

* Hewing et al., Learning-Based Model Predictive Control: Toward Safe Learning in Control. Annual Review of Control, Robotics, and Autonomous Systems, 2020.

Outline of the talk

1. Learning set-valued models from data

   Makdesi, Girard & Fribourg, Data-driven models of monotone systems,
   IEEE Transactions on Automatic Control, 2023.

2. Safe learning-based nonlinear model predictive control

   Makdesi, Girard & Fribourg, Safe learning-based model predictive control using the compatible models approach.
   European Journal of Control, 2023.

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Learning set-valued models – formulation

Assume the unknown map g satisfies the following property*

$$\forall x \preccurlyeq x', \forall u \preccurlyeq u', A(x' - x) + B(u' - u) \preccurlyeq g(x', u') - g(x, u)$$

where A and B are known matrices.

Given the data set

$$D = \{(x_i, u_i, x_i^+) |\, x_i^+ = f(x_i, u_i) + g(x_i, u_i), i = 1, \ldots, N\}$$

Compute the "tightest" set-valued map $G: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ such that

$$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, g(x, u) \in G(x, u)$$

* True if g is Lipschitz or if g has lower bounded derivatives

Reformulation using monotone maps (1)

- Consider the map $h: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ given by

$$h(x, u) = g(x, u) - Ax - Bu$$

- h is unknown and monotone

$$\forall x \preccurlyeq x', \forall u \preccurlyeq u', h(x, u) \preccurlyeq h(x', u')$$

- Consider the modified data set $D' = \{(x_i, u_i, y_i) |\ i = 1, \ldots, N\}$ where

$$y_i = h(x_i, u_i) = x_i^+ - f(x_i, u_i) - Ax_i - Bu_i$$

Reformulation using motonone maps (2)

- A map $\tilde{h} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is consistent with D' (i.e. $\tilde{h} \in C_{D'}$) if $\tilde{h}$ is monotone and

$$\forall i = 1, \dots, N, y_i = \tilde{h}(x_i, u_i)$$

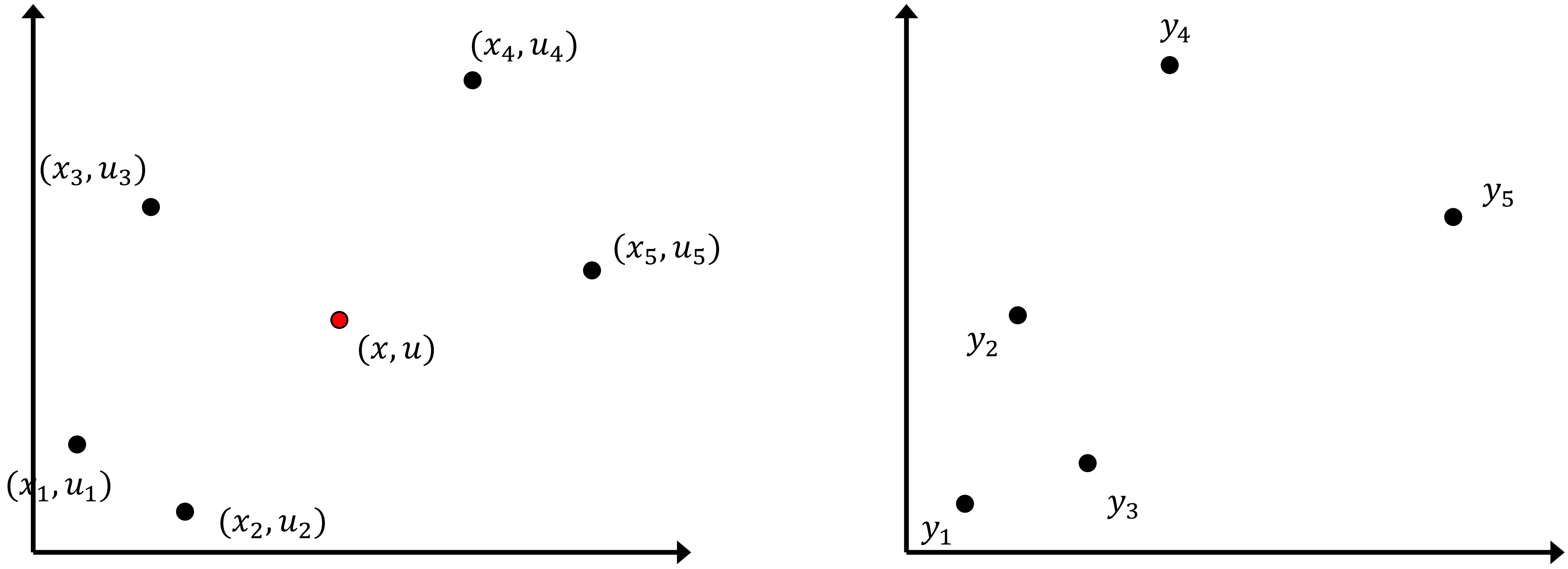
- A set-valued map $H: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is simulating D' (i.e. $H \in S_{D'}$) if for all $\tilde{h} \in C_{D'}$

$$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, \tilde{h}(x, u) \in H(x, u)$$

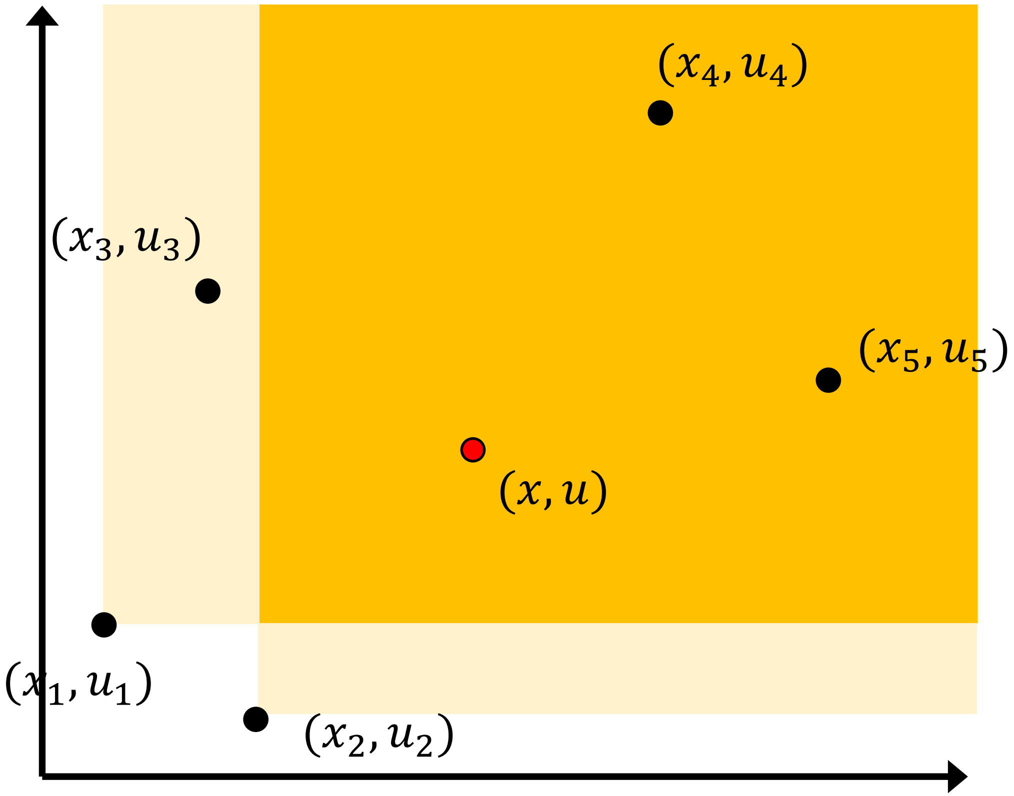
- It is minimal if for all $\widetilde{H} \in S_{D'}$

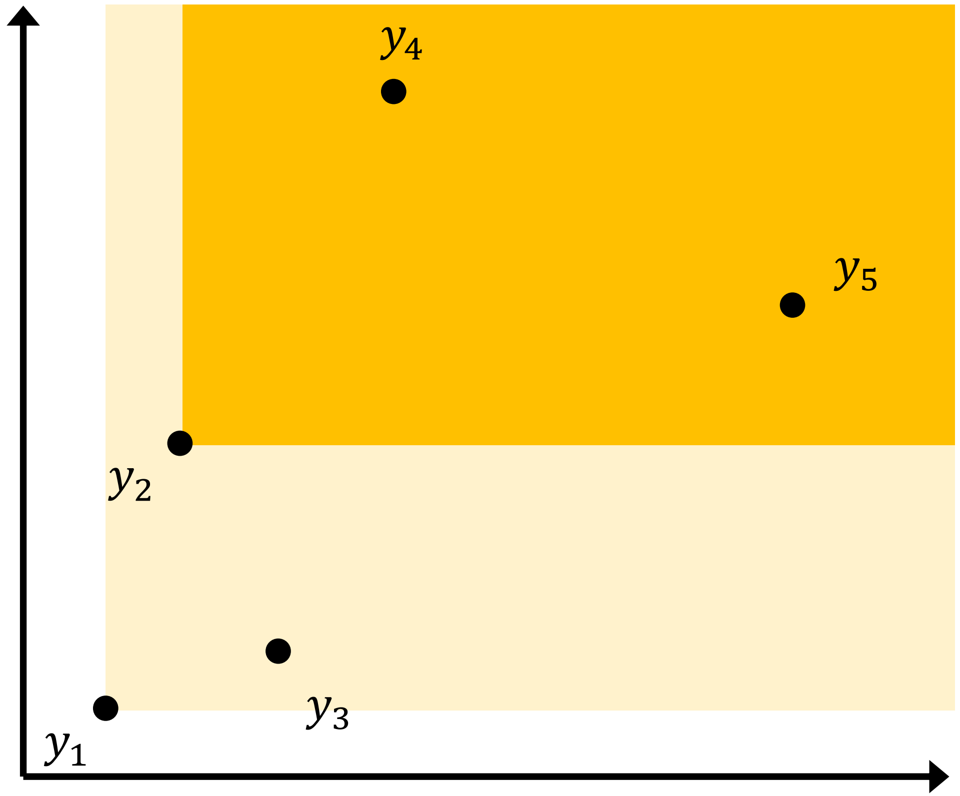
$$\forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m, H(x, u) \subseteq \widetilde{H}(x, u)$$

Computation of the minimal simulating map

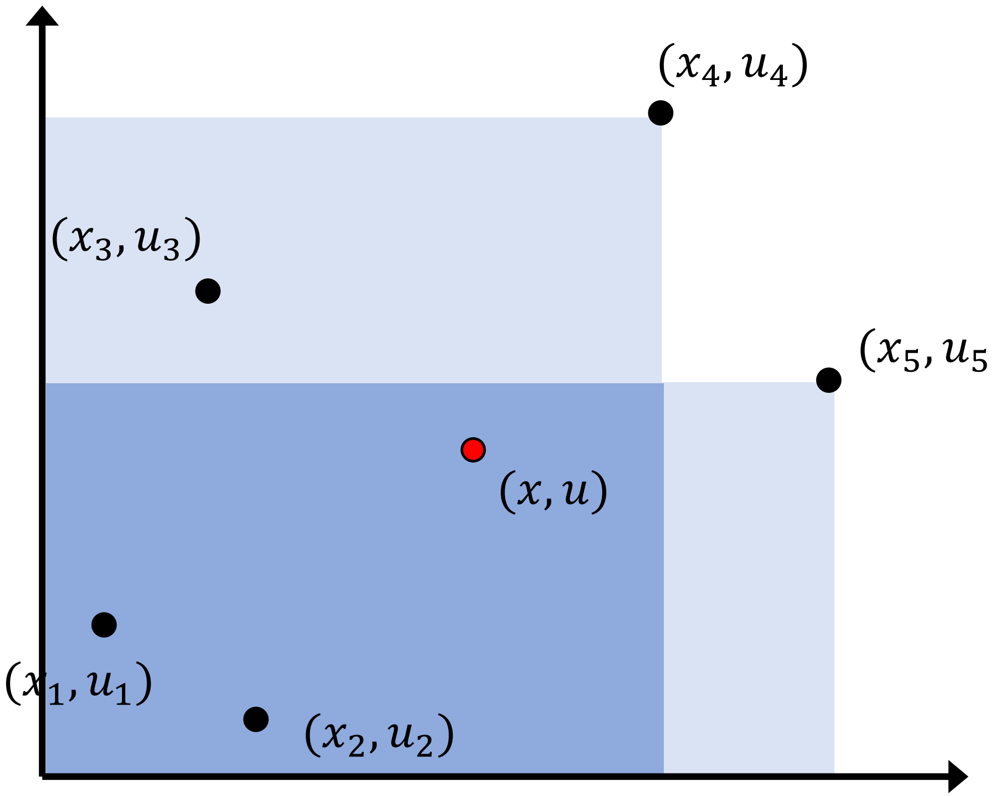


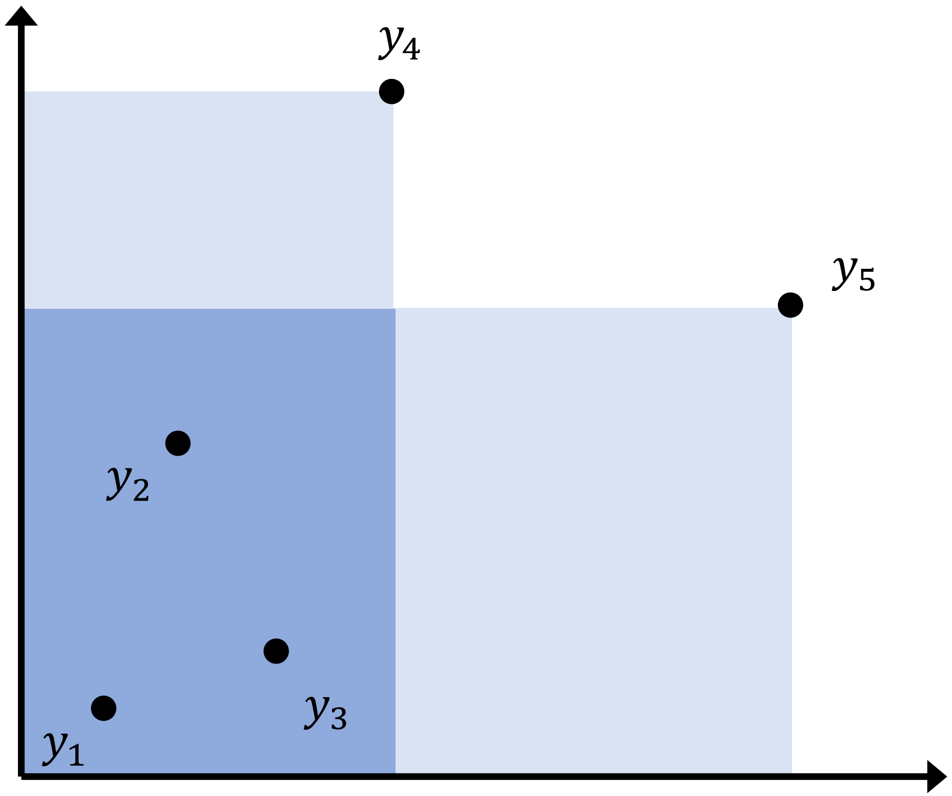
Computation of the minimal simulating map





Computation of the minimal simulating map





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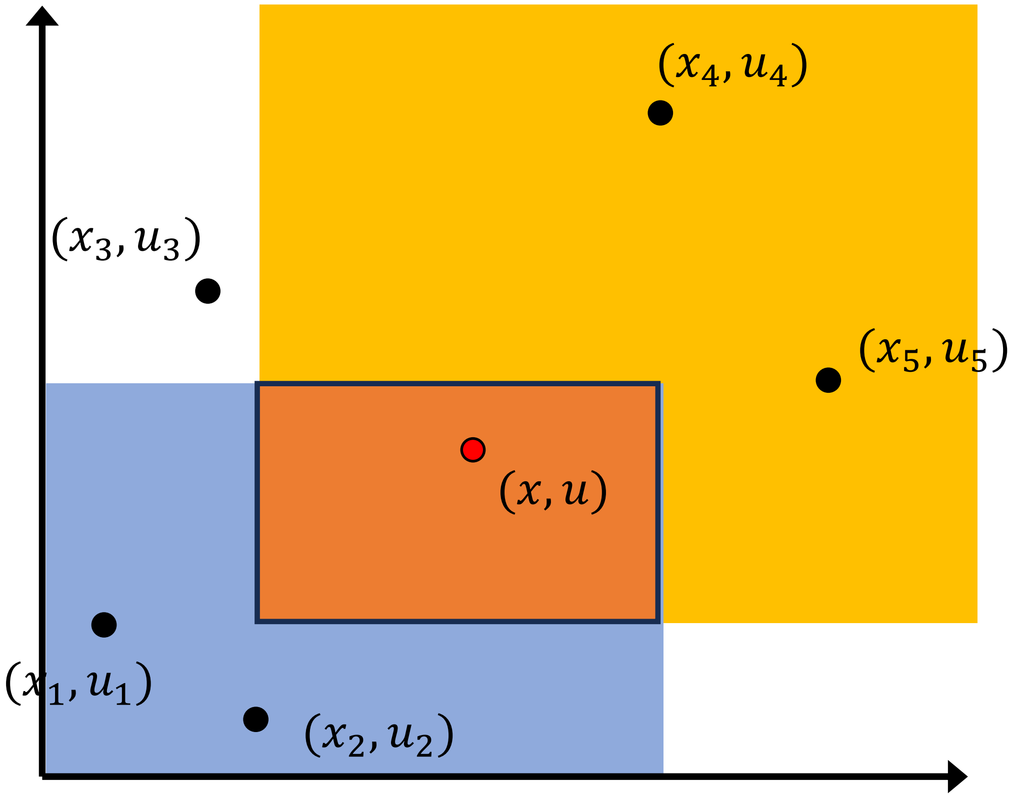

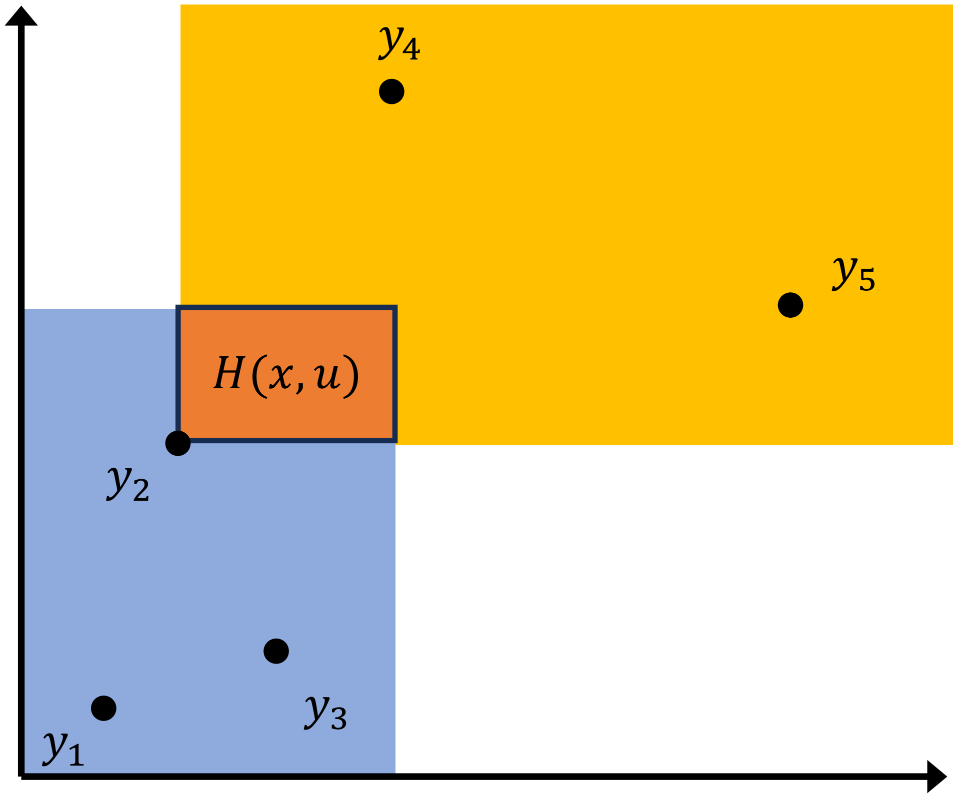

Minimal simulating map – theorem

For a data set D', there exists a unique minimal simulating map $H: \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$.

It satisfies the following properties:

1. It is inner-semi continuous
2. For all $x \in \mathbb{R}^n, u \in \mathbb{R}^m, H(x,u)$ is an interval of $\mathbb{R}^n$
3. There exist interval partitions $\left(X_q\right)_{q \in Q}$ and $\left(U_p\right)_{p \in P}$ of $\mathbb{R}^n$ and $\mathbb{R}^m$, and a collection of intervals $\left(Y_{q,p}\right)_{q \in Q, p \in P}$ such that

$$H(x,u) = \bigcap_{\{(q,p) \mid x \in cl(X_q), u \in cl(U_p)\}} Y_{q,p}$$

Effective implementation

Computation of the minimal simulating map:

- Computational complexity: $\mathcal{O}(N \times \log(|Q| \times |P|) + |Q| \times |P|)$
- With $|Q| \times |P| = (N+1)^{n+m}$, we get $\mathcal{O}(N^{n+m})$, the complexity is polynomial in the size of the data set.

Fix the partitions $\left(X_q\right)_{q \in Q}$ and $\left(U_p\right)_{p \in P}$ a priori:

- Still "safe" but introduces some conservatism: H minimal in the class of simulating maps piecewise constant on these partitions.
- The complexity becomes linear in the size of the data set

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Data-driven safety filter

- We consider the data-driven difference inclusion given by:

$$x_{t+1} \in \tilde{f}(x_t, u_t) + H(x_t, u_t), x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m$$

  where $\tilde{f}(x, u) = f(x, u) + Ax + Bu$.

- Given state and input safety constraints $\mathbb{X}$ and $\mathbb{U}$, we want to compute a robust controlled invariant set $\mathbb{X}_s \subseteq \mathbb{X}$ such that

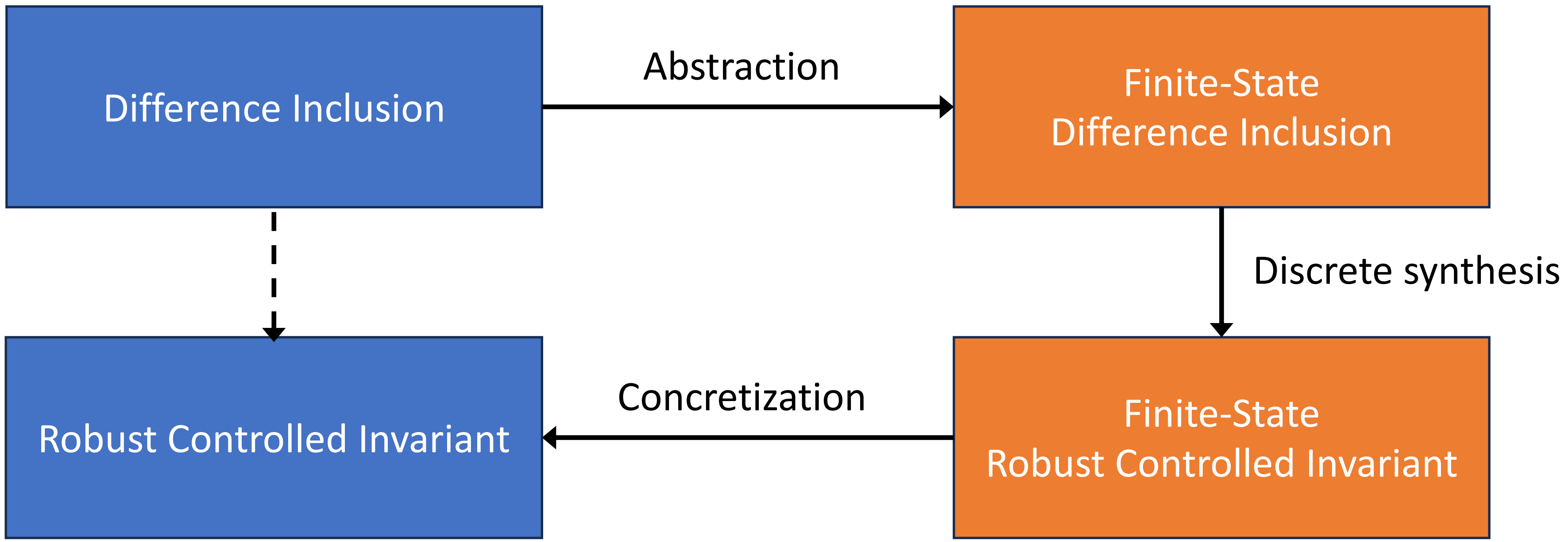
$$\forall x \in \mathbb{X}_s, \exists u \in \mathbb{U}, \tilde{f}(x, u) + H(x, u) \subseteq \mathbb{X}_s$$

- Then a safety filter is given by the set-valued map $C_s: \mathbb{X}_s \rightrightarrows \mathbb{U}$

$$C_s(x) = \left\{ u \in \mathbb{U} \middle| \, \tilde{f}(x, u) + H(x, u) \subseteq \mathbb{X}_s \right\}$$

Symbolic control approach

A robust controlled invariant set can be computed using the symbolic control approach*:

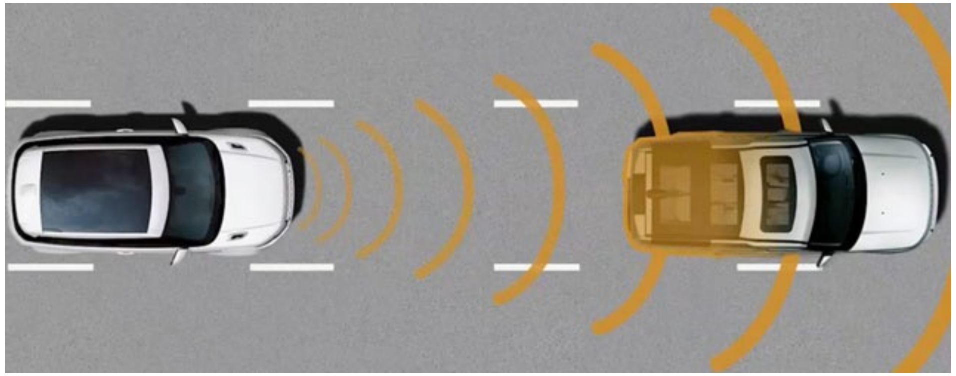


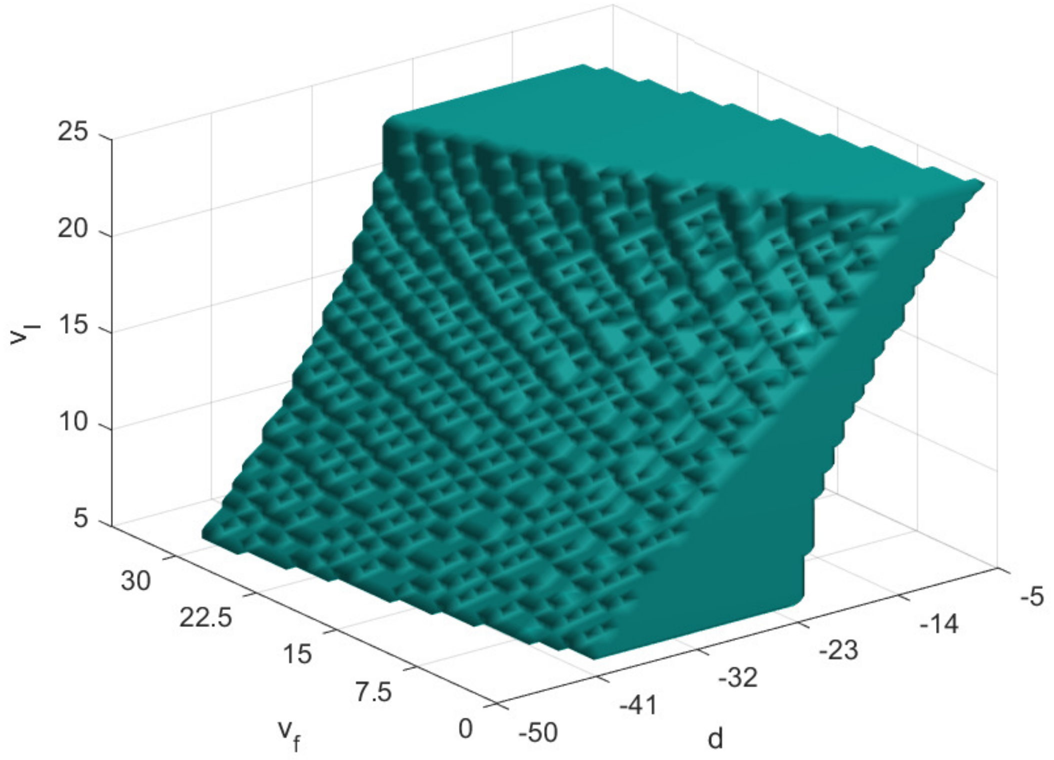
*Girard, Meyer & Saoud, Approches symboliques pour le contrôle des systèmes non linéaires. *Techniques de l'Ingénieur*, 2024.

Example – adaptive cruise control

Consider two vehicles (leader and follower):

- Relative distance d
- Follower and leader velocity v_1 and v_2
- Unknown dynamics





Robust controlled invariant set computed from 10^6 data points.

Safe learning-based MPC – the compatible models approach

We consider the following MPC:

$$\underset{u_{0|t}\,,\,\dots\,,\,u_{r-1|t}}{\text{minimize}} \quad \sum_{k=0}^{r-1} l_k\big(x_{k|t}, u_{k|t}\big) + l_r\big(x_{r|t}\big)$$

$$u_{k|t} \in \mathbb{U}, x_{k+1|t} \in \mathbb{X}, \qquad k = 0, \dots, r-1$$ constraints

$$u_{0|t} \in C_s(x_{0|t})$$ data-driven safety filter

$$x_{k+1|t} = \tilde{f}\big(x_{k|t}, u_{k|t}\big) + \tilde{h}\big(x_{k|t}, u_{k|t}\big)$$ data-driven prediction

where $\tilde{h}$: $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous selection* of H:

$$\forall x \in \mathbb{R}^n, \forall\, u \in \mathbb{R}^m, \tilde{h}(x,u) \in H(x,u)$$

* A continuous selection of H exists by Michael selection theorem.

The compatible models approach – theorem

Consider the unknown discrete-time nonlinear system:

$$x_{t+1} = f(x_t, u_t) + g(x_t, u_t), x_t \in \mathbb{R}^n, u_t \in \mathbb{R}^m$$

interconnected with the safe learning-based MPC with

$$x_{0|t} = x_t, u_t = u_{0|t}, \forall t \in \mathbb{N}$$

Then, the optimization problem is recursively feasible and

$$x_t \in \mathbb{X}, u_t \in \mathbb{U}, \forall t \in \mathbb{N}$$

Effective construction of the continuous selection

- Select values at the vertices $(x_v, u_v)_{v \in V}$ of the partition $\left(X_q \times U_p\right)_{q \in Q, p \in P}$

$$\forall v \in \mathrm{V}, \tilde{h}(x_v, u_v) \in H(x_v, u_v)$$

- Then interpolate in each cell of the partition by a multi-affine function (multi-variate polynomial of degree 1 in each variable) :

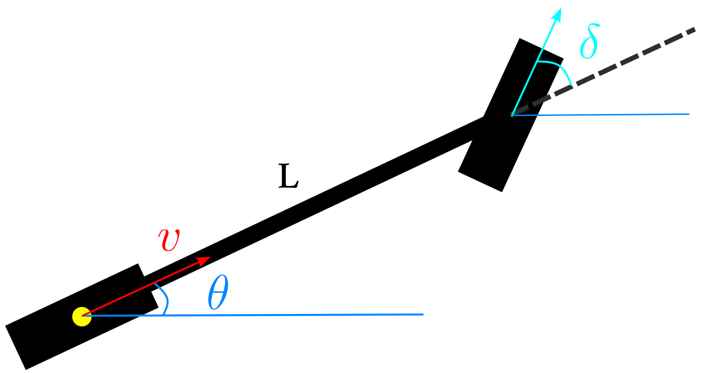
$$\forall (x, u) \in X_q \times U_p$$

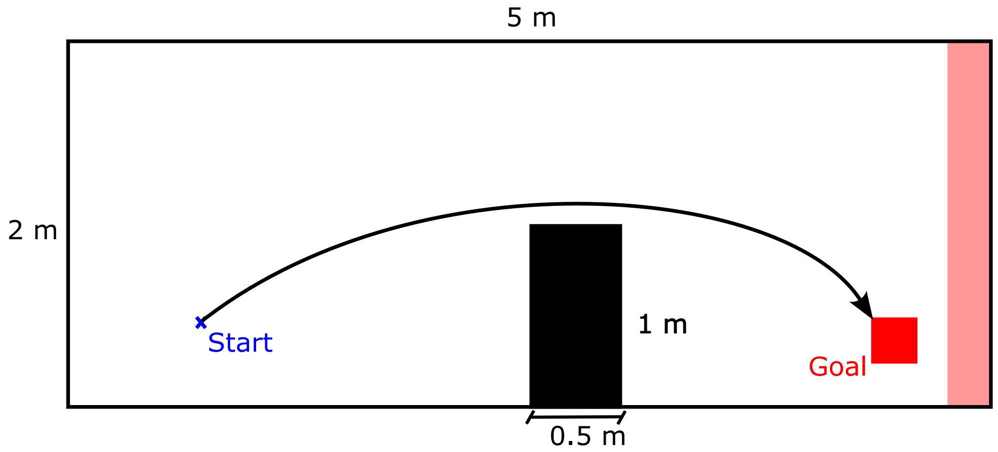
$$\tilde{h}(x, u) = a_0 + a_1 x_1 + a_2 x_2 + a_3 u_1 + a_4 x_1 x_2 + a_5 x_1 u_1 + a_6 x_2 u_1 + a_7 x_1 x_2 u_1 + \cdots$$

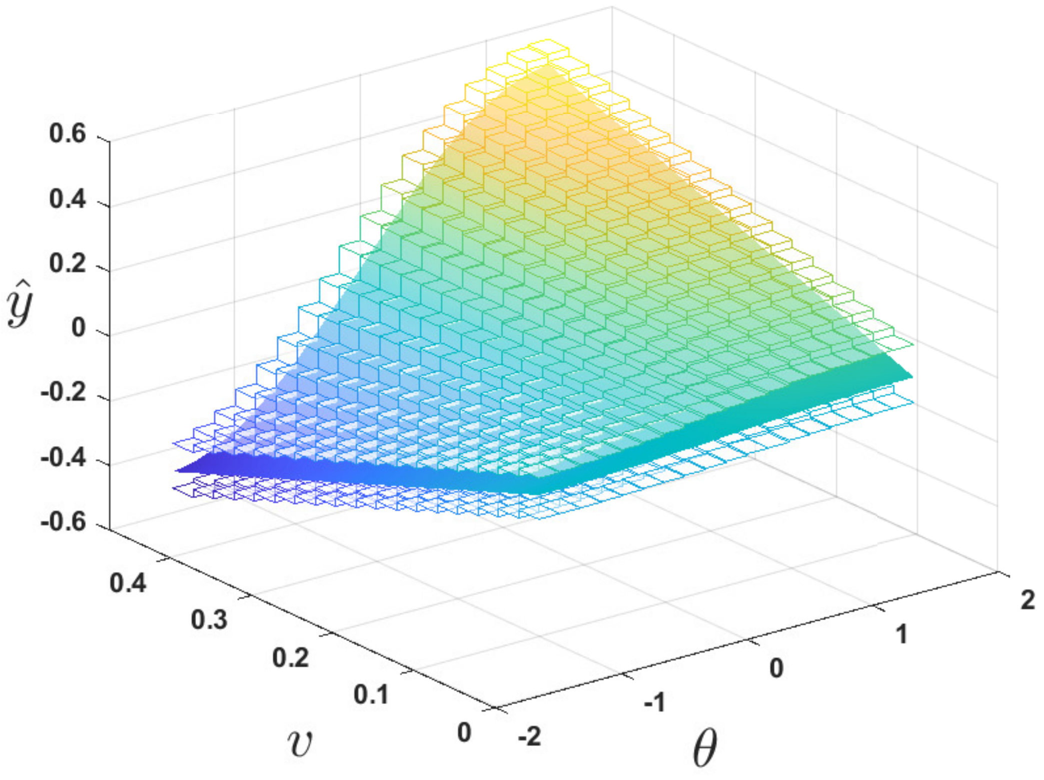
- From properties of multi-affine maps, $\tilde{h}$ is a continuous selection of H.

Example – bicycle model with disturbances

3 states, 2 inputs

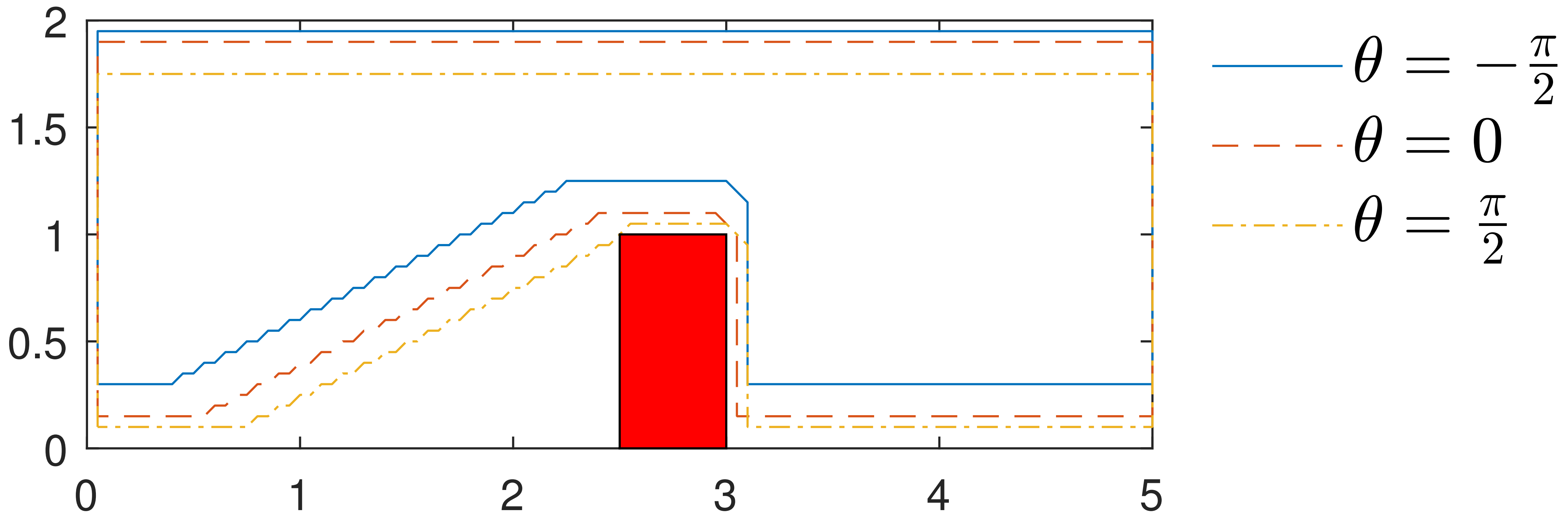






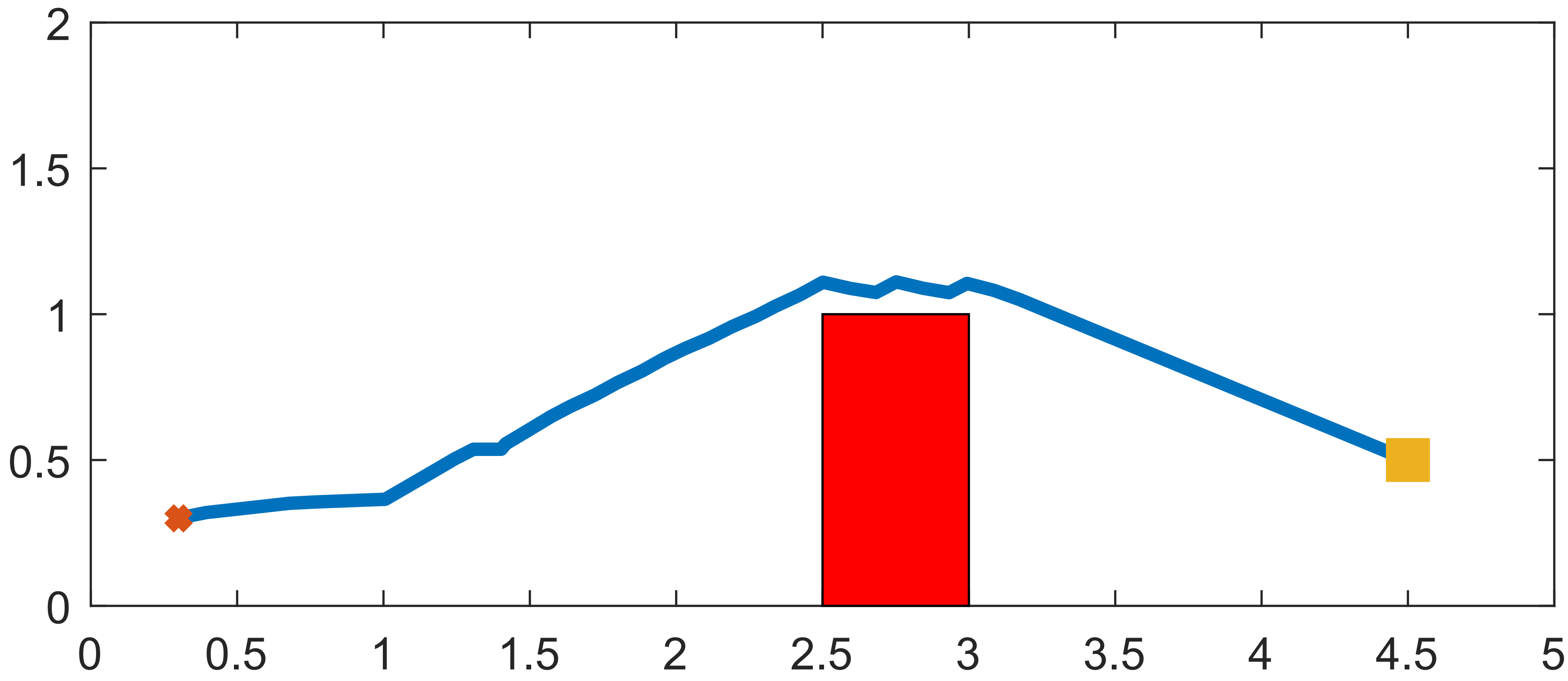
Simulating map

Example – bicycle model with disturbances



Robust control invariant set

Example – bicycle model with disturbances



Trajectory using safe learning-based MPC

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Online learning – model update

Consider two data sets D and D' and the associated minimal simulating maps H_D and $H_{D'}$, then

$$H_{D \cup D'}(x,u) = H_D(x,u) \cap H_{D'}(x,u), \forall x \in \mathbb{R}^n, \forall u \in \mathbb{R}^m$$

- The minimal simulating map can be updated from newly collected data D' without reprocessing older data D.
- Afterwards, the continuous selection can be updated by adapting the value at the vertices of the partition to the new constraints.

Online learning – safety filter update

Update the robust controlled invariant set:

- Check if some states outside the old invariant can be controlled to the reach the invariant (computationally cheap, conservative)
- Synthesize a new robust controlled invariant set from scratch using the updated model (computationally expensive, no conservatism)

The safety filter is easily updated given the new robust controlled invariant set.

Conclusion and outlook

A set-valued approach to safe-learning:

- Results grounded in theory of monotone maps
- Computational approach based on combination of symbolic control and MPC
- Formal safety guarantees

Future research directions:

- Improvement of the MPC implementation (warm start, non-smooth constraints)
- Efficient online learning (active learning, dual control…)
- Safe learning of time-varying systems (handling outdated data)
- Physics-informed learning (e.g. unknown map is solution of a PDE)