The Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality

... from tensor to multivariate rational approximation and more

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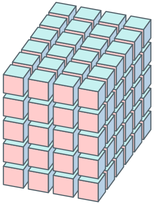
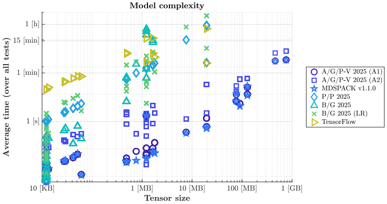
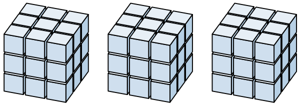
Workshop on data-driven control and analysis of dynamical systems

https://arxiv.org/abs/2405.00495 [in SIAM Review - Research Spotlight]
https://arxiv.org/abs/2506.04791 [extensive benchmark]
https://github.com/cpoussot/mLF [research code package]





$$\mathbf{H}({}^1x,\cdots,{}^8x)=\mathbf{H}(r_w,r,T_u,H_u,T_l,H_l,L,K_w)=\frac{2\pi T_u(H_u-H_l)}{\ln\left(\frac{r}{r_w}\right)\left(1+\frac{2LT_u}{\ln(r/r_w)r_w^2K_w}\right)+\frac{T_u}{T_l}}$$

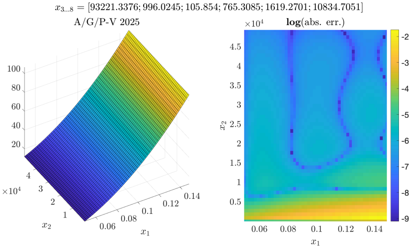


$$\begin{array}{ccccc} {}^1x & \times & \cdots & \times & {}^8x \\ [\underline{r_w},\overline{r_w}] & \times & \cdots & \times & [\underline{K_w},\overline{K_w}] \end{array}$$

$$\textbf{tab}_8 \in \mathbb{C}^{8\times 8\times\cdots\times 8}$$

≈130 `Mo ('real')`

#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.	max err.
30	A1	1e-09,1	1.02e+04	19.3	0.00455	2e-09	0.061
	A2	1e-15,2	1.02e+04	39.1	0.00456	2.93e-09	0.0611



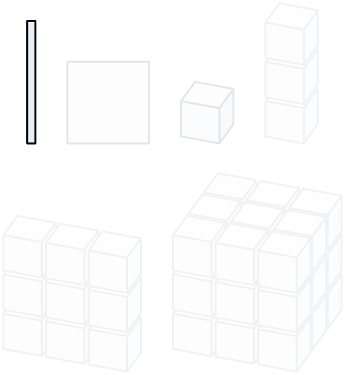
S. Surjanovic, *"Borehole function"*, `https://www.sfu.ca/~ssurjano/borehole.html`.

Column / Row data

$$ {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \Big\} \xrightarrow{\mathbf{H}({}^1x)} \Big\{ \mathbf{w}_{j_1}, \mathbf{v}_{i_1} $$

${}^1x$	
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1}$
${}^1\mu_{1,\cdots,q_1}$	$\mathbf{V}_{q_1}$

Tensors (1-D) tab_1



Forewords

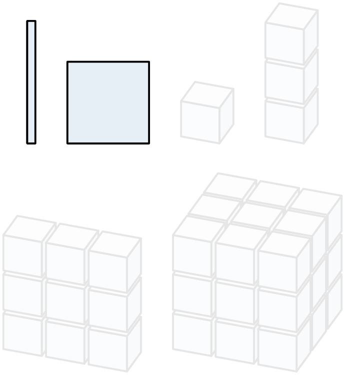
Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1x, {}^2x)} \left\{ \mathbf{w}_{j_1,j_2}, \mathbf{v}_{i_1,i_2} \right.$$

${}^1x \backslash {}^2x$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1,k_2}$	ϕ_{cr}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rc}	$\mathbf{V}_{q_1,q_2}$

Tensors (2-D) tab$_2$



Forewords

Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^1\mathbf{x} = {}^1\lambda_{j_1}, {}^1\mu_{i_1} \\ {}^2\mathbf{x} = {}^2\lambda_{j_2}, {}^2\mu_{i_2} \\ {}^3\mathbf{x} = {}^3\lambda_{j_3}, {}^3\mu_{i_3} \end{array}\right\} \xrightarrow{\mathbf{H}({}^1x, {}^2x, {}^3x)} \left\{ \mathbf{w}_{j_1,j_2,j_3}, \mathbf{v}_{i_1,i_2,i_3} \right.$$

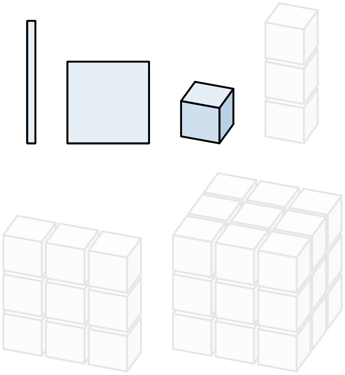
$${}^3x = {}^3\lambda_{1,\cdots,k_3}$$

${}^2x$ / ${}^1x$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	$\mathbf{W}_{k_1,k_2,k_3}$	ϕ_{crc}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rcc}	ϕ_{rrc}

$${}^3x = {}^3\mu_{1,\cdots,q_3}$$

${}^2x$ / ${}^1x$	${}^2\lambda_{1,\cdots,k_2}$	${}^2\mu_{1,\cdots,q_2}$
${}^1\lambda_{1,\cdots,k_1}$	ϕ_{crr}	ϕ_{crr}
${}^1\mu_{1,\cdots,q_1}$	ϕ_{rcr}	$\mathbf{V}_{q_1,q_2,q_3}$

Tensors (3-D) tab$_3$



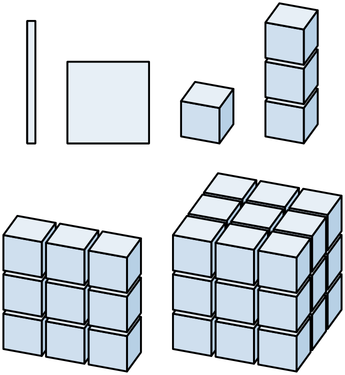
Forewords

Data (and tensors)

Column / Row data

$$\left.\begin{array}{l} {}^{1}\mathbf{x} = {}^{1}\lambda_{j_1}, {}^{1}\mu_{i_1} \\ {}^{2}\mathbf{x} = {}^{2}\lambda_{j_2}, {}^{2}\mu_{i_2} \\ {}^{3}\mathbf{x} = {}^{3}\lambda_{j_3}, {}^{3}\mu_{i_3} \\ \quad\vdots \\ {}^{n}\mathbf{x} = {}^{n}\lambda_{j_n}, {}^{n}\mu_{i_n} \end{array}\right\} \xrightarrow{\mathbf{H}({}^{1}x,\cdots,{}^{n}x)} \left\{ \mathbf{w}_{j_1,\cdots,j_n}, \mathbf{v}_{i_1,\cdots} \right.$$

Tensors (n-D) tab$_n$



Forewords

Contributions claim & trajectory of the presentation

List of contributions

- n-D tensor data to n-D Loewner matrix $\mathbb{L}_n$
- n-variable transfer functions
- Taming the curse of dimensionality
  - » in computation effort (`flop`)
  - » in storage needs (`Bytes`)
  - » in accuracy
- n-variable **decoupling**
  - » **KST** formulation for rational functions
  - » connection with **KAN**
- Comparison with **MLP, KAN, AAA**

arXiv

A.C. Antoulas, I-V. Gosea and C. P-V, *"On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality"*, SIAM Review, 2025, `https://arxiv.org/abs/2405.00495`.

A.C. Antoulas, I-V. Gosea, C. P-V. and P. Vuillemin, *"Tensor-based multivariate function approximation: methods benchmarking and comparison"*, `https://arxiv.org/abs/2506.04791`.

Multi-variate data, function & Loewner matrix

1-D case

$$\begin{cases} P_c^{(1)} & := & \left\{ \left({}^1\lambda_{j_1}; \mathbf{w}_{j_1} \right),\ j_1 = 1, \dots, k_1 \right\} \\ P_r^{(1)} & := & \left\{ \left({}^1\mu_{i_1}; \mathbf{v}_{i_1} \right),\ i_1 = 1, \dots, q_1 \right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1,j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{{}^1\mu_{i_1} - {}^1\lambda_{j_1}}$$

Lagrangian form

$$\mathbf{g}({}^1x) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1}\mathbf{w}_{j_1}}{{}^1x - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{{}^1x - {}^1\lambda_{j_1}}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

Multi-variate data, function & Loewner matrix

1-D case (example)

Data generated from $\mathbf{H}({}^1x) = \mathbf{H}(s) = (s^2+4)/(s+1)$ of complexity (2)

$$\left.\begin{array}{lcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [2,4,6,8] \end{array}\right\} \xrightarrow{\mathbf{H}} \left\{\begin{array}{lcl} \mathbf{w}_{j_1} & = & [5/2, 13/4, 29/6] \\ \mathbf{v}_{i_1} & = & [8/3, 4, 40/7, 68/9] \end{array}\right.$$

Loewner matrix

$$\mathbb{L}_1 = \begin{bmatrix} \frac{1}{6} & \frac{7}{12} & \frac{13}{18} \\ \frac{1}{2} & \frac{3}{4} & \frac{5}{6} \\ \frac{9}{14} & \frac{23}{28} & \frac{37}{42} \\ \frac{13}{18} & \frac{31}{36} & \frac{49}{54} \end{bmatrix}$$

Null space

$$\mathbf{c}_1 = \begin{bmatrix} \frac{1}{3} \\ -\frac{4}{3} \\ 1 \end{bmatrix}$$

Lagrangian form

$$\mathbf{g}(s) = \frac{\frac{5}{6\,(s-1)} - \frac{13}{3\,(s-3)} + \frac{29}{6\,(s-5)}}{\frac{1}{3\,(s-1)} - \frac{4}{3\,(s-3)} + \frac{1}{s-5}} = \mathbf{H}(s)$$

Multi-variate data, function & Loewner matrix

2-D case

$$\begin{cases} P_c^{(2)} & := & \left\{({}^1\lambda_{j_1}, {}^2\lambda_{j_2}; \mathbf{w}_{j_1,j_2}),\ j_1 = 1,\dots,k_1 \quad j_2 = 1,\dots,k_2\right\} \\ P_r^{(2)} & := & \left\{({}^1\mu_{i_1}, {}^2\mu_{i_2}; \mathbf{v}_{i_1,i_2}),\ i_1 = 1,\dots,q_1 \quad i_2 = 1,\dots,q_2\right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell_{j_1,j_2}^{i_1,i_2} = \frac{\mathbf{v}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})({}^2\mu_{i_2} - {}^2\lambda_{j_2})}$$

Lagrangian form

$$\mathbf{g}({}^1x, {}^2x) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{({}^1x - {}^1\lambda_{j_1})({}^2x - {}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{({}^1x - {}^1\lambda_{j_1})({}^2x - {}^2\lambda_{j_2})}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_2) = \mathcal{N}(\mathbb{L}_2)$$

$$\mathbf{c}_2 = \left[\begin{array}{c} c_{1,1} \\ \vdots \\ c_{1,k_2} \\ \hline \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{array}\right] \in \mathbb{C}^{k_1 k_2}$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

$$
\left.\begin{array}{ccc}
{}^1\lambda_{j_1} & = & [1,3,5] \\
{}^1\mu_{i_1} & = & [0,2,4] \\
{}^2\lambda_{j_2} & = & [-1,-3] \\
{}^2\mu_{i_2} & = & [-2,-4]
\end{array}\right\} \xrightarrow{\mathbf{H}}
\left[\begin{array}{cc|cc}
-\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\
-\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\
-\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\
\hline
0 & 0 & 0 & 0 \\
-1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\
-\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9}
\end{array}\right]
$$

Loewner matrix

$$
\mathbb{L}_2 = \left[\begin{array}{cc|cc|cc}
\frac{1}{3} & -\frac{3}{5} & \frac{3}{5} & -\frac{9}{7} & \frac{5}{7} & -\frac{5}{3} \\
\frac{1}{9} & \frac{3}{5} & \frac{1}{5} & \frac{9}{7} & \frac{5}{21} & \frac{5}{3} \\
\frac{19}{15} & -1 & \frac{1}{5} & -\frac{79}{35} & \frac{23}{35} & -\frac{101}{45} \\
\hline
\frac{41}{63} & \frac{59}{35} & -\frac{17}{105} & \frac{11}{7} & \frac{1}{7} & \frac{127}{63} \\
\frac{89}{63} & -\frac{139}{105} & \frac{97}{35} & -\frac{5}{7} & -1 & -\frac{79}{21} \\
\frac{61}{81} & \frac{293}{135} & \frac{239}{135} & \frac{205}{63} & -\frac{223}{189} & \frac{11}{9}
\end{array}\right]
$$

Null space

$$
\mathbf{c}_2 = \left[\begin{array}{c}
-\frac{1}{3} \\
\frac{5}{9} \\
\hline
\frac{10}{9} \\
-\frac{14}{9} \\
\hline
-\frac{7}{9} \\
1
\end{array}\right]
$$

Multi-variate data, function & Loewner matrix

2-D case (example)

Data generated from $\mathbf{H}({}^1x,{}^2x)=\mathbf{H}(s,t)=(s^2t)/(s-t+1)$ of complexity $(2,1)$

$$\left.\begin{array}{rcl} {}^1\lambda_{j_1} & = & [1,3,5] \\ {}^1\mu_{i_1} & = & [0,2,4] \\ {}^2\lambda_{j_2} & = & [-1,-3] \\ {}^2\mu_{i_2} & = & [-2,-4] \end{array}\right\} \xrightarrow{\mathbf{H}} \left[\begin{array}{cc|cc} -\frac{1}{3} & -\frac{3}{5} & -\frac{1}{2} & -\frac{2}{3} \\ -\frac{9}{5} & -\frac{27}{7} & -3 & -\frac{9}{2} \\ -\frac{25}{7} & -\frac{25}{3} & -\frac{25}{4} & -10 \\ \hline 0 & 0 & 0 & 0 \\ -1 & -2 & -\frac{8}{5} & -\frac{16}{7} \\ -\frac{8}{3} & -6 & -\frac{32}{7} & -\frac{64}{9} \end{array}\right]$$

Lagrangian form

$$\mathbf{g}(s,t)=-\frac{\frac{1}{9(s-1)(t+1)}-\frac{1}{3(s-1)(t+3)}-\frac{2}{(s-3)(t+1)}+\frac{6}{(s-3)(t+3)}+\frac{25}{9(s-5)(t+1)}-\frac{25}{3(s-5)(t+3)}}{\frac{1}{3(s-1)(t+1)}-\frac{5}{9(s-1)(t+3)}-\frac{10}{9(s-3)(t+1)}+\frac{14}{9(s-3)(t+3)}+\frac{7}{9(s-5)(t+1)}-\frac{1}{(s-5)(t+3)}}=\mathbf{H}(s,t)$$

Multi-variate data, function & Loewner matrix

n-D case

$$\begin{cases} P_c^{(n)} := \left\{({}^1\lambda_{j_1}, {}^2\lambda_{j_2}, \cdots, {}^n\lambda_{j_n}; \mathbf{w}_{j_1,j_2,\cdots,j_n}),\ j_l = 1,\dots,k_l, \quad l = 1,\cdots,n\right\} \\ P_r^{(n)} := \left\{({}^1\mu_{i_1}, {}^2\mu_{i_2}, \cdots, {}^n\mu_{i_n}; \mathbf{v}_{i_1,i_2,\cdots,i_n}),\ i_l = 1,\dots,q_l, \quad l = 1,\cdots,n\right\} \end{cases}$$

Loewner matrix

$$\mathbb{L}_n \in \mathbb{C}^{q_1 q_2 \cdots q_n \times k_1 k_2 \cdots k_n}$$

$$\ell_{j_1,j_2,\cdots,j_n}^{i_1,i_2,\cdots,i_n} = \frac{\mathbf{v}_{i_1,i_2,\cdots,i_n} - \mathbf{w}_{j_1,j_2,\cdots,j_n}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})\cdots({}^n\mu_{i_n} - {}^n\lambda_{j_n})}$$

Lagrangian form

$$\mathbf{g}({}^1x, \cdots, {}^nx) = \frac{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n} \mathbf{w}_{j_1,\cdots,j_n}}{({}^1x - {}^1\lambda_{j_1})\cdots({}^nx - {}^n\lambda_{j_n})}}{\sum_{j_1=1}^{k_1} \cdots \sum_{j_n=1}^{k_n} \frac{c_{j_1,\cdots,j_n}}{({}^1x - {}^1\lambda_{j_1})\cdots({}^nx - {}^n\lambda_{j_n})}}$$

Null space

$$\textbf{span}\ (\mathbf{c}_n) = \mathcal{N}(\mathbb{L}_n)$$

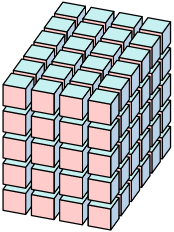
$$\mathbf{c}_n = \left[\begin{array}{c} c_{1,\cdots,1} \\ \vdots \\ c_{1,\cdots,k_n} \\ \hline \vdots \\ \hline c_{k_1,\cdots,1} \\ \vdots \\ c_{k_1,\cdots,k_n} \end{array}\right] \in \mathbb{C}^{k_1 \cdots k_n}$$

Taming the curse of dimensionality

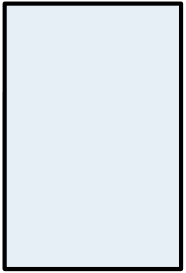
n-variable Loewner matrix operator

$$\begin{array}{ccc} \mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \ldots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1)\times\cdots\times(k_n+q_n)} & \longrightarrow & \mathbb{C}^{Q\times K} \\ \left({}^1\lambda_{j_1}, {}^1\mu_{i_1}, \ldots, {}^n\lambda_{j_n}, {}^n\mu_{i_n}, \textbf{tab}_n\right) & \longmapsto & \mathbb{L}_n \end{array}$$

n-D tensor $\textbf{tab}_n$



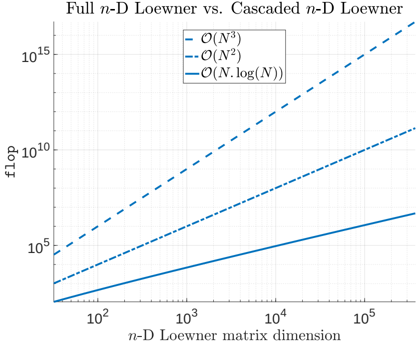
matrix $\mathbb{L}_n$



Taming the curse of dimensionality

Null space `flop` and memory issues

log-log scale



(rows) $Q = q_1 q_2 \dots q_n$ and
(columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD `flop` estimation is

- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

Storage issue

Note that $Q \times K$ matrix storage estimation is

- in real double $QK \frac{8}{2^{20}}$ `MB`
- in complex double $2QK \frac{8}{2^{20}}$ `MB`

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

${}^1x \backslash {}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1x$, for ${}^2x = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2x$ for ${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$
  $\left[\; \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \;\right]^\top$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2 t)/(s-t+1)$ of complexity $(2,1)$

${}^1x \backslash {}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{bmatrix}$$

- 1 $\mathbb{L}_1$ along ${}^1x$, for ${}^2x = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2x$ for ${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top = \left[\; \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \;\right]^\top$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

${}^1x \backslash {}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along ${}^1x$, for ${}^2x = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2x$ for ${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix} \text{ and } \mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\begin{array}{ccc} \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 & \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 & \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3 \end{array}\right]^\top$$

Taming the curse of dimensionality

2-D case (example cont'd, to get the idea)

Data generated from $\mathbf{H}({}^1x, {}^2x) = \mathbf{H}(s,t) = (s^2t)/(s-t+1)$ of complexity $(2,1)$

${}^1x \backslash {}^2x$	${}^2\lambda_1 = -1$	${}^2\lambda_2 = -3$	${}^2\mu_1 = -2$	${}^2\mu_2 = -4$
${}^1\lambda_1 = 1$	$h_{1,1} = -\frac{1}{3}$	$h_{1,2} = -\frac{3}{5}$	$h_{1,3} = -\frac{1}{2}$	$h_{1,4} = -\frac{2}{3}$
${}^1\lambda_2 = 3$	$h_{2,1} = -\frac{9}{5}$	$h_{2,2} = -\frac{27}{7}$	$h_{2,3} = -3$	$h_{2,4} = -\frac{9}{2}$
${}^1\lambda_3 = 5$	$h_{3,1} = -\frac{25}{7}$	$h_{3,2} = -\frac{25}{3}$	$h_{3,3} = -\frac{25}{4}$	$h_{3,4} = -10$
${}^1\mu_1 = 0$	$h_{4,1} = 0$	$h_{4,2} = 0$	$h_{4,3} = 0$	$h_{4,4} = 0$
${}^1\mu_2 = 2$	$h_{5,1} = -1$	$h_{5,2} = -2$	$h_{5,3} = -\frac{8}{5}$	$h_{5,4} = -\frac{16}{7}$
${}^1\mu_3 = 4$	$h_{6,1} = -\frac{8}{3}$	$h_{6,2} = -6$	$h_{6,3} = -\frac{32}{7}$	$h_{6,4} = -\frac{64}{9}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_2)} \mathbf{c}_2 = \left[\begin{array}{c} -\frac{1}{3} \\ \frac{5}{9} \\ \hline \frac{10}{9} \\ -\frac{14}{9} \\ \hline -\frac{7}{9} \\ 1 \end{array}\right]$$

- 1 $\mathbb{L}_1$ along ${}^1x$, for ${}^2x = {}^2\lambda_2 = -3$
- 3 $\mathbb{L}_1$ along ${}^2x$ for ${}^1x = \{{}^1\lambda_1, {}^1\lambda_2, {}^1\lambda_3\}$
- Scaled null space $\mathbf{c}_2^\top =$

$$\left[\ \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 \quad \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \quad \mathbf{c}_1^{{}^1\lambda_3} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_3\ \right]^\top$$

$$\mathbf{c}_1^{{}^2\lambda_2} = \begin{bmatrix} \frac{5}{9} \\ -\frac{14}{9} \\ 1 \end{bmatrix}$$ and $$\mathbf{c}_1^{{}^1\lambda_1} = \begin{bmatrix} -\frac{3}{5} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_2} = \begin{bmatrix} -\frac{5}{7} \\ 1 \end{bmatrix}, \mathbf{c}_1^{{}^1\lambda_3} = \begin{bmatrix} -\frac{7}{9} \\ 1 \end{bmatrix}$$

Theorem: 2-D to 1-D

Being given the tableau $\textbf{tab}_2$ tensor in response of the 2-variables $\mathbf{H}({}^1x, {}^2x)$ function, the null space of the corresponding 2-D Loenwer matrix $\mathbb{L}_2$, is spanned by

$$\mathcal{N}(\mathbb{L}_2) = \mathbf{vec}\left[\mathbf{c}_1^{{}^2\lambda_1} \cdot \left[\mathbf{c}_1^{{}^1\lambda_{k_1}}\right]_1, \cdots, \mathbf{c}_1^{{}^2\lambda_{k_2}} \cdot \left[\mathbf{c}_1^{{}^1\lambda_{k_1}}\right]_{k_2}\right],$$

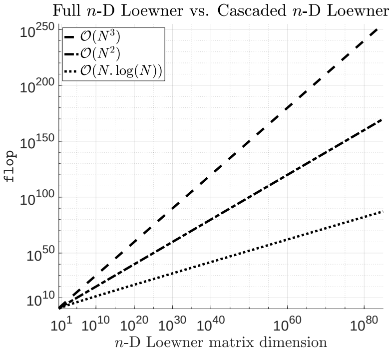
where

- $\mathbf{c}_1^{{}^1\lambda_{k_1}} = \mathcal{N}(\mathbb{L}_1^{{}^1\lambda_{k_1}})$,
  i.e. the null space of the **1-D Loewner matrix** for frozen ${}^1x = {}^1\lambda_{k_1}$, and
- $\mathbf{c}_1^{{}^2\lambda_{j_2}} = \mathcal{N}(\mathbb{L}_1^{{}^2\lambda_{j_2}})$,
  i.e. the j_1-th null space of the **1-D Loewner matrices** for frozen ${}^2x = \{{}^2\lambda_1, \cdots, {}^2\lambda_{k_2}\}$.

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



(rows) $Q = q_1 q_2 \dots q_n$ and
(columns) $K = k_1 k_2 \dots k_n$

$$\mathbb{L}_n \in \mathbb{C}^{Q \times K}$$

Computational issue

Note that $Q \times K$ matrix SVD `flop` estimation is

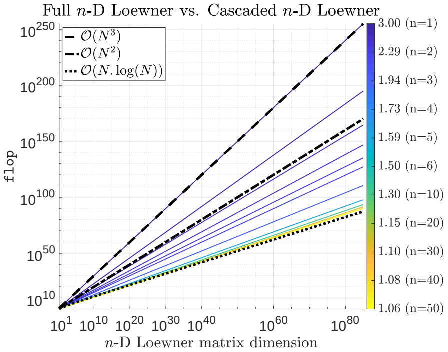
- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

$\Rightarrow$ **The CURSE of dimensionality**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Theorem: Recursive complexity

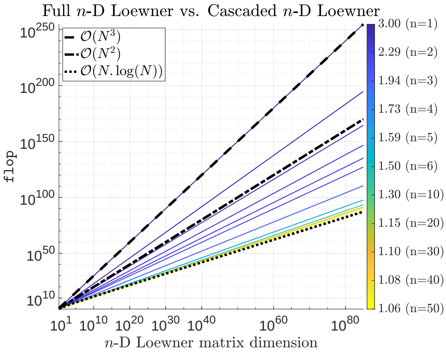
$$\texttt{flop}_1(n) = \sum_{j=1}^{n} \left(k_j^3 \prod_{l=1}^{j} k_{l-1} \right) \text{ where } k_0 = 1.$$

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

Taming the curse of dimensionality

Null space - `flop` complexity

log-log scale



Corollary: Worst case complexity

k interpolation points per variables.

$$\overline{\texttt{flop}_1} = k^3 \frac{1-k^n}{1-k} = k^3 \frac{1-N}{1-k},$$

which is a (n finite) geometric series of ratio k.

$\Rightarrow$ **The CURSE of dimensionality is TAMED**

$$\begin{array}{llll} \mathcal{O}(N^3) & \rightarrow & \mathcal{O}(N^{2.29}) & \text{for } n=2 \\ & \rightarrow & \mathcal{O}(N^{1.94}) & \text{for } n=3 \\ & \vdots & & \\ & \rightarrow & \mathcal{O}(N^{1.5}) & \text{for } n=6 \\ & \vdots & & \\ & \rightarrow & \mathcal{O}(N^{1.06}) & \text{for } n=50 \end{array}$$

Taming the curse of dimensionality

Null space - memory

With similar importance, the data storage is a key element in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}} \prod_{l}^{n} q_l + k_l \quad \texttt{MB} \text{ (example tableau } 2 \cdot [20, 6, 4, 6, 8, 2] = 2 \cdot [k_1, k_2, k_3, k_4, k_5, k_6] \text{ needs } 45 \texttt{ MB)}$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \quad \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 GB
flop: $9.78 \cdot 10^{13}$

Taming the curse of dimensionality

Null space - memory

With similar importance, the data storage is a key element in the curse of dimensionality. The data (tableau) storage is (complex and double precision)

$$\frac{8}{2^{20}} \prod_{l}^{n} q_l + k_l \ \ \texttt{MB} \text{ (example tableau } 2 \cdot [20,6,4,6,8,2] = 2 \cdot [k_1,k_2,k_3,k_4,k_5,k_6] \text{ needs } 45\ \texttt{MB})$$

Full n-D Loewner

Construction of

$$\mathbb{L}_n \in \mathbb{C}^{N \times N}$$

where $N = k_1 k_2 \cdots k_n$, needs

$$\frac{8}{2^{20}} N^2 \ \ \texttt{MB}$$

Example: $N = 46,080$
Memory: 31.64 GB
`flop`: $9.78 \cdot 10^{13}$

Cascaded n-D Loewner

Construction of

$$\mathbb{L}_1 \in \mathbb{C}^{\overline{k} \times \overline{k}}$$

where $\overline{k} = \max_j k_j$, needs

$$\frac{8}{2^{20}} \overline{k}^2 \ \ \texttt{MB}$$

Example: $\overline{k} = 20$
Memory: 6.25 KB
`flop`: $8.13 \cdot 10^5$

Variables decoupling, KST and KANs

Loewner, Kolmogorov Superposition Theorem and Hilbert's 13th problem

Variable decoupling

Given data $\mathbf{tab}_n$, the latter achieves variables decoupling, and the null space can be equivalently written as:

$$\mathbf{c}_n = \underbrace{\mathbf{c}^{{}^n x}}_{\mathbf{Bary}({}^n x)} \odot \underbrace{(\mathbf{c}^{{}^{n-1} x} \otimes \mathbf{1}_{k_n})}_{\mathbf{Bary}({}^{n-1} x)} \odot \underbrace{(\mathbf{c}^{{}^{n-2} x} \otimes \mathbf{1}_{k_n k_{n-1}})}_{\mathbf{Bary}({}^{n-2} x)} \odot \cdots \odot \underbrace{(\mathbf{c}^{{}^1 x} \otimes \mathbf{1}_{k_n \dots k_2})}_{\mathbf{Bary}({}^1 x)}.$$

where $\mathbf{c}^{{}^l x}$ denotes the vectorized barycentric coefficients related to the l-th variable.

"Kolmogorov proved that every continuous function of several variables can be represented as a superposition of continuous functions of one variable and the operation of addition (1957). Thus, it is as if there are no functions of several variables at all. Seriously speaking, Kolmogorov's theorem is a brilliant example of his mastery. In particular, the theorem shows that Hilbert's conjecture (to it's 13th problem) is wrong."

A.G. Vitushkin, *"On Hilbert's thirteenth problem and related questions"*, Russian Math. Surveys 59:1, pp. 11-25.

Variables decoupling, KST and KANs

KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1x \cdot {}^2x$)

$$
\begin{aligned}
{}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
{}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x+1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{({}^1x+1.0)({}^2x-1.0)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x+1.0)} \\
1.0 & 1.0 & 1.0 & \frac{1}{({}^1x-1.0)({}^2x-1.0)}
\end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{c}^{{}^2x} &= \textbf{vec}\begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\
\mathbf{c}^{{}^1x} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot (\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2})
$$

$$
\mathbf{D} = \begin{pmatrix}
\mathbf{c}^{{}^1x}\cdot\mathbf{Lag}({}^1x) & \mathbf{c}^{{}^2x}\cdot\mathbf{Lag}({}^2x) \\
-\frac{1.0}{{}^1x+1.0} & -\frac{1.0}{{}^2x+1.0} \\
-\frac{1.0}{{}^1x+1.0} & \frac{1}{{}^2x-1.0} \\
\frac{1}{{}^1x-1.0} & -\frac{1.0}{{}^2x+1.0} \\
\frac{1}{{}^1x-1.0} & \frac{1}{{}^2x-1.0}
\end{pmatrix}
$$

Equivalent denominator and numerator read:

$$
\sum_{i\text{-th row}} \prod_{j\text{-th col}} [\mathbf{D}]_{i,j} \text{ and } \sum_{i\text{-th row}} \mathbf{w} \cdot \prod_{j\text{-th col}} [\mathbf{D}]_{i,j}
$$

Variables decoupling, KST and KANs

KANs via Loewner with rational activation functions ($\mathbf{H} = {}^1x \cdot {}^2x$)

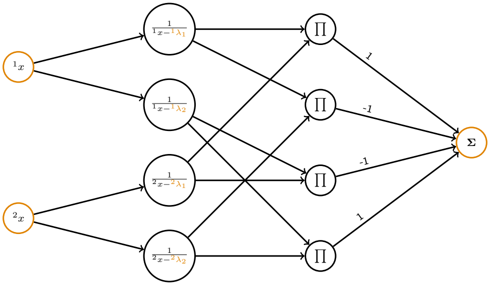
$$
\begin{aligned}
{}^1\lambda_{j_1} &= \begin{pmatrix} -1 & 1 \end{pmatrix} \\
{}^2\lambda_{j_2} &= \begin{pmatrix} -1 & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix}
\mathbf{c} & \mathbf{w} & \mathbf{c}\cdot\mathbf{w} & \mathbf{Lag} \\
1.0 & 1.0 & 1.0 & \frac{1}{\left({}^1x+1.0\right)\left({}^2x+1.0\right)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{\left({}^1x+1.0\right)\left({}^2x-1.0\right)} \\
-1.0 & -1.0 & 1.0 & \frac{1}{\left({}^1x-1.0\right)\left({}^2x+1.0\right)} \\
1.0 & 1.0 & 1.0 & \frac{1}{\left({}^1x-1.0\right)\left({}^2x-1.0\right)}
\end{pmatrix}
$$

$$
\begin{aligned}
\mathbf{c}^{{}^2x} &= \textbf{vec}\begin{pmatrix} -1.0 & -1.0 \\ 1.0 & 1.0 \end{pmatrix} \\
\mathbf{c}^{{}^1x} &= \begin{pmatrix} -1.0 \\ 1.0 \end{pmatrix}
\end{aligned}
$$

$$
\mathbf{c}_2 = \mathbf{c}^{{}^2x} \odot \left(\mathbf{c}^{{}^1x} \otimes \mathbf{1}_{k_2}\right)
$$

Denominator Network view



Other methods

Rat. app (B/G 2025)

- p-AAA

KAN (P/P 2025)

- Kolmogorov Arnold Network

MLP (TensorFlow 2025)

- Multi Layer Perceptron
- Keras is the high-level API of TensorFlow (by Google) `https://www.tensorflow.org/guide/keras?hl=en`
- Dense connected network / ReLU activation / ADAM optim. / 1000 iterations / random init.

`TensorFlow interface` (Python code)

```python
import numpy as np
import math
import matplotlib.pyplot as plt
import tensorflow as tf
from keras.models import Sequential
from matplotlib import cm

# Data
def H(x):
    #y = pow(x[:,0],2)*x[:,1]
    #y = 1/2*(x[:,0] + np.abs(x[:,0])) + 1/10*x[:,1] #1
    #y = x[:,0]*x[:,1] #3
    #y = np.exp(x[:,0]*x[:,1])/( (pow(x[:,0],2)-1.44)*(pow(x[:,1],2)-1.44) ) #6
    #y = np.tanh(4*(x[:,0]-x[:,1])) #8
    #y = pow(np.abs((x[:,0]-x[:,1])),3) #10
    y = (pow(x[:,0],2) + pow(x[:,1],2) + x[:,0] - x[:,1] + 1) / (pow(x[:,0],3)  + pow(x[:,1],2) + 4) #15
    return np.transpose(np.array([y]))

n   = 2
N1  = 40
N2  = 40
x1  = np.linspace(-1, 1, N1)
x2  = np.linspace(-1, 1, N2)

# IP
N   = N1*N2
tab = np.zeros((N,1))
```

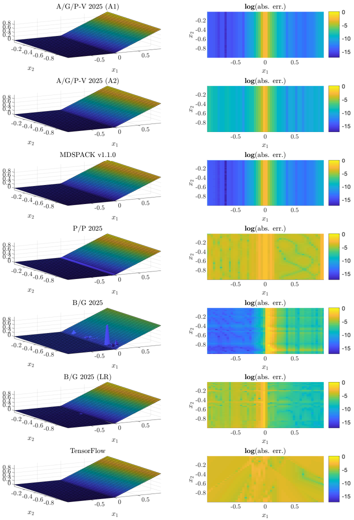

L. Balicki and S. Gugercin, *"Multivariate Rational Approximation via Low-Rank Tensors and the p-AAA Algorithm"*, SISC, 2025.

M. Poluektov and A. Polar, *"Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method"*, `https://arxiv.org/abs/2305.08194`.

M. Abadi et al., *"TensorFlow: Large-scale machine learning on heterogeneous systems, 2015"*, Software available from `tensorflow.org`.

Comparisons

Irrational & rational functions (example #1)



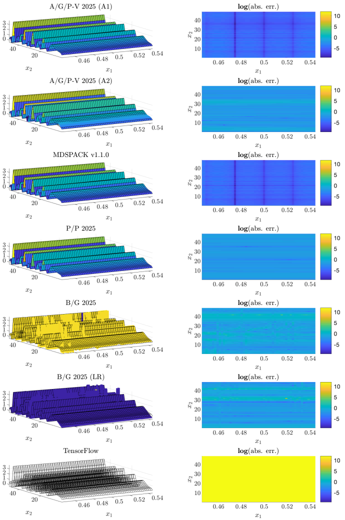
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
1	A/G/P-V 2025 (A1)	1e-11,3	144	**0.0171**	0.000699	**1.39e − 17**
	A/G/P-V 2025 (A2)	1e-15,3	160	0.0823	**0.000389**	4.49e-13
	MDSPACK v1.1.0	1e-11,1e-06	144	0.0623	0.000699	2.43e-17
	P/P 2025	1,0.95,50,0.01,4,6,9	**130**	0.282	0.0017	3.08e-07
	B/G 2025	0.001,20	612	0.353	0.0288	3.75e-16
	B/G 2025 (LR)	0.001,20,4	480	0.678	0.00147	7.82e-12
	TensorFlow		257	14.6	0.00074	6.31e-08

$$\mathrm{ReLU}({}^{1}x) + \frac{1}{100}{}^{2}x$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: 12.5 **KB** (40^2 points)
- Bounds: $\begin{pmatrix} -1 & 1 \end{pmatrix} \times \begin{pmatrix} -1 & -10^{-10} \end{pmatrix}$

Comparisons

Irrational & rational functions (example #34)



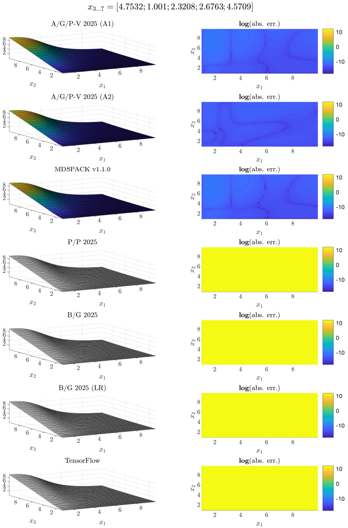
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
34	A/G/P-V 2025 (A1)	1e-10,2	2.32e+03	1.5	4.86e-05	**2.74e −**
	A/G/P-V 2025 (A2)	1e-15,2	864	**0.654**	0.137	1.51e-06
	MDSPACK v1.1.0	1e-10,1e-09	2.3e+03	1.58	**3.21e − 05**	1.87e-08
	P/P 2025	1,0.95,50,0.01,10,12,21	**676**	79	0.0254	1.36e-05
	B/G 2025	0.001,20	1.22e+03	85.6	2.53	0.000305
	B/G 2025 (LR)	1e-06,20,3	1.36e+03	16.9	4.74	1.23e-05
	TensorFlow	NaN	NaN	NaN	NaN	NaN

$$\mathrm{Re}(\zeta(^1x + \imath^2x))$$

- Riemann ζ function (real part), [none]
- Domain: $\mathbb{R}$
- Tensor size: 1.22 **MB** (400^2 points)
- Bounds: $\begin{pmatrix} \frac{9}{20} & \frac{11}{20} \end{pmatrix} \times \begin{pmatrix} 1 & 50 \end{pmatrix}$

Comparisons

Irrational & rational functions (example #43)



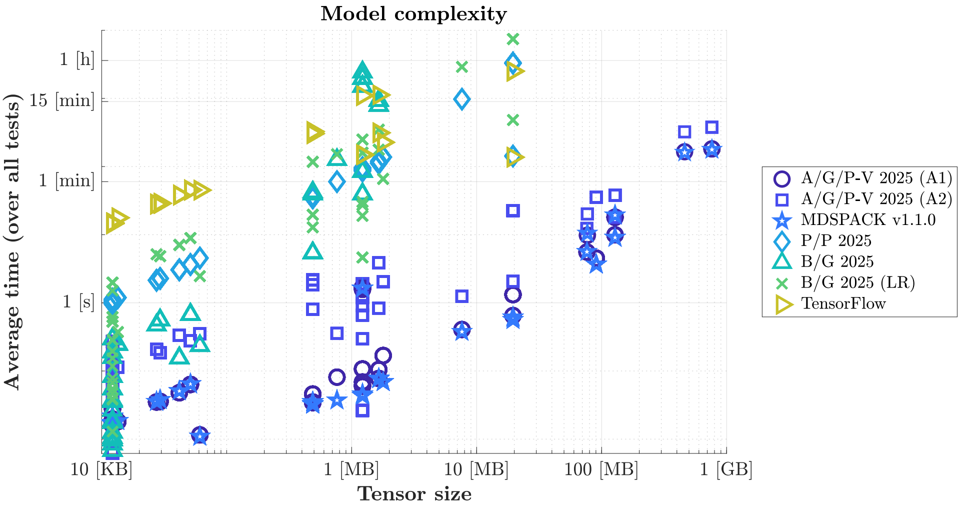
#	Alg.	Parameters	Dim.	CPU [s]	RMSE	min err.
43	A/G/P-V 2025 (A1)	0.0001,1	**1.73e + 04**	**5.58**	1.41e-12	1.76e-16
	A/G/P-V 2025 (A2)	1e-15,1	1.73e+04	12.5	**2.39e − 13**	**1.21e − 17**
	MDSPACK v1.1.0	0.0001,1e-06	1.73e+04	5.72	1.4e-12	1.49e-16
	P/P 2025	NaN	NaN	NaN	NaN	NaN
	B/G 2025	NaN	NaN	NaN	NaN	NaN
	B/G 2025 (LR)	NaN	NaN	NaN	NaN	NaN
	TensorFlow	NaN	NaN	NaN	NaN	NaN

$$\frac{{}^3x\ {}^2x^3+1}{{}^1x^4+{}^2x^2\ {}^3x+{}^4x^2+{}^5x+{}^6x^3+{}^7x}$$

- Reference: Personal communication, [none]
- Domain: $\mathbb{R}$
- Tensor size: $\mathbb{R}$
- Tensor size: 76.3 **MB** (10^7 points)
- Bounds: $\begin{pmatrix} 1 & 10 \end{pmatrix}^7$

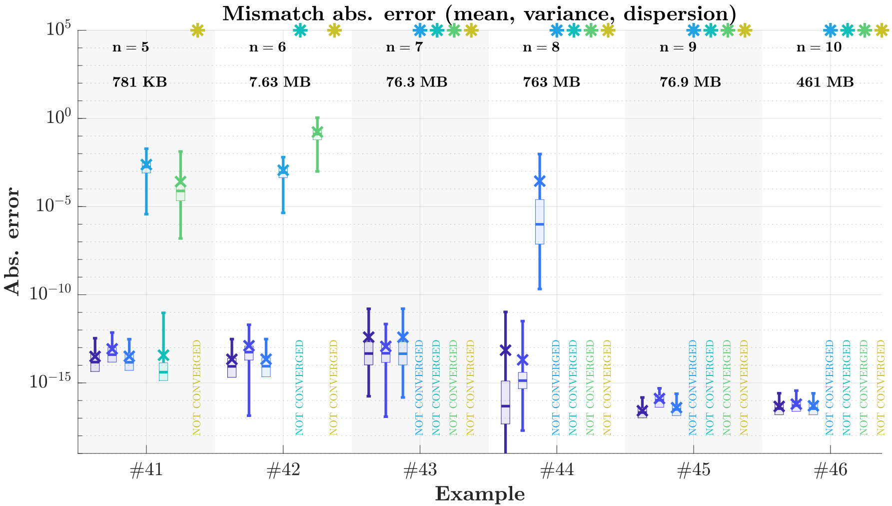
Comparisons

Irrational & rational functions (time, scalability)



Comparisons

Irrational & rational functions (accuracy, scalability)



Conclusion

Take home message

Main contributions

From any n-th order multi-variate transfer function / data tensor

- Construct a transfer function in barycentric form
- **Construct a realization with controlled complexity**
- **Tame the computational complexity**
- Two algorithms (direct & iterative)
- Connection with Kolmogorov theorem
- Connection with Kolmogorov networks

Side effects

[Sci. con.] Tensor rank approximation

[Sci. con.] Achieve multi-linearization of NEVP

[Sci. con.] Exact (Loewner) matrix null space computation

[Dyn. sys.] Multi-variate / parametric realization

Collaboration with
A.C. Antoulas [Rice Univ.]
I.V. Goșea [MPI]
P. Vuillemin [ONERA]

https://arxiv.org/abs/2405.00495
https://arxiv.org/abs/2506.04791
https://github.com/cpoussot/mLF
https://cpoussot.github.io

In parting... if enough time

Numerical examples, 20-D example

$$\mathbf{H}({}^1x, {}^2x, \ldots, {}^{20}x) =$$

$$\frac{3 \cdot {}^1x^3 + 4 \cdot {}^8x + {}^{12}x + {}^{13}x \cdot {}^{14}x + {}^{15}x}{{}^1x + {}^2x^2 \cdot {}^3x + {}^4x + {}^5x + {}^6x + {}^7x \cdot {}^8x + {}^9x \cdot {}^{10}x \cdot {}^{11}x + {}^{13}x + {}^{13}x^3 \cdot \pi + {}^{17}x + {}^{18}x \cdot {}^{19}x - {}^{20}x}$$

Statistics

- 20-D tensor of dimension (≥ 48 TB in real double precision)
- Complexity: $(3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 1, 1, 1, 1)$
- n-D Loewner matrix $6,291,456^2 \rightarrow 288$ TB of storage in real double precision
- Full SVD: $2.49 \cdot 10^{20}$ `flop`
  Recursive SVD: $5.43 \cdot 10^{7}$ `flop`
- error $\approx 10^{-11}$

In parting... if enough time

Numerical examples, 20-D example

$$\mathbf{H}({}^1x, {}^2x, \ldots, {}^{20}x) =$$

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  Recursive SVD: $5.43 \cdot 10^7$ `flop` $\rightarrow 5.03 \cdot 10^7$ `flop`
- error $\approx 10^{-11}$

In parting... if enough time

Numerical examples (from 2 to 20 variables)

#4 Rational function

$$s_4{}^3 + \frac{s_1\, s_3}{s_3{}^2 + s_1 + s_2 + 1}$$

#5 Rational function

$$\frac{s_3{}^2 + s_1\, s_3\, s_5{}^3}{s_1{}^3 + s_4 + s_2\, s_3}$$

#6 Rational function

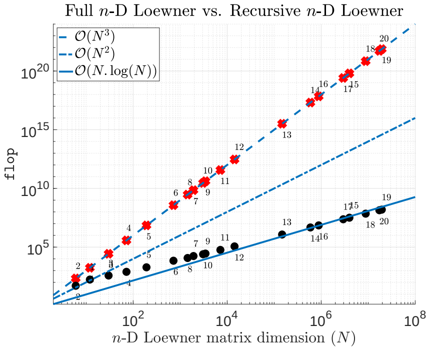
$$\frac{-\sqrt{2}\, s_6{}^2 + s_1 + s_3}{s_1{}^2 + s_4{}^3 + s_5{}^2 + s_6 + s_2\, s_3}$$

#7 Rational function

$$\frac{s_3\, s_2{}^3 + 1}{s_3\, s_2{}^2 + s_4{}^2 + s_6{}^3 + s_1 + s_5 + s_7}$$

#19 Rational function

$$\frac{3\, s_1{}^3 + s_{18}{}^2 + 4\, s_8 + s_{12} + s_{15} + s_{13}\, s_{14}}{s_3\, s_2{}^2 + \pi\, s_{16}{}^3 + s_{17}{}^2 + s_1 + s_4 + s_5 + s_6 + s_{13} + s_{19} + s_7\, s_8 + s_9\, s_{10}\, s_{11}}$$



In parting... if enough time

Numerical examples (rational and irrational)

#16 Arc-tangent function

$$\frac{\operatorname{atan}(x_1) + \operatorname{atan}(x_2) + \operatorname{atan}(x_3) + \operatorname{atan}(x_4)}{{x_1}^2\,{x_2}^2 - {x_1}^2 - {x_2}^2 + 1}$$

#17 Exponential function

$$\frac{e^{x_1\,x_2\,x_3\,x_4}}{{x_1}^2 + {x_2}^2 - x_3\,x_4 + 3}$$

#18 Sinc function

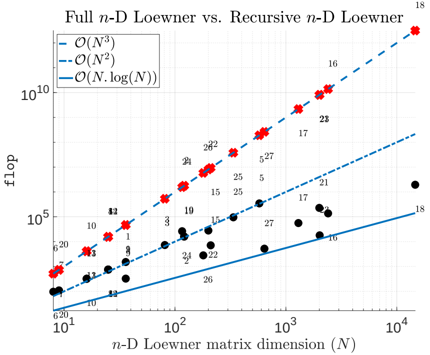
$$\frac{10\,\sin(x_1)\,\sin(x_2)\,\sin(x_3)\,\sin(x_4)}{x_1\,x_2\,x_3\,x_4}$$

#19 Sinc function

$$\frac{10\,\sin(x_1)\,\sin(x_2)}{x_1\,x_2}$$

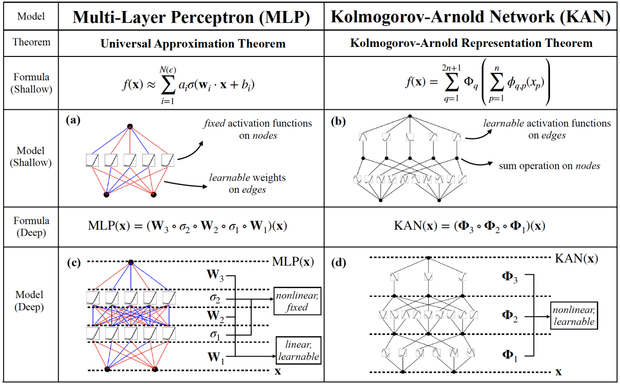
#20 Polynomial function

$${x_1}^2 + x_1\,x_2 + {x_2}^2 - x_2 + 1$$



KANs features

- **Inspired by the Kolmogorov-Arnold representation theorem**
- The model output is a composition of **sums** and **learnable activation functions** (e.g. splines)
- Alternate to Multi-Layer Perceptrons (MLP), having fixed activation functions (e.g. ReLU), **inspired by the universal approximation theorem**



Comparison between MLP and KAN (figure from *Z. Liu et al.*)

Z. Liu et al., *"KAN: Kolmogorov–Arnold Networks"*, `https://arxiv.org/abs/2404.19756`.

In parting... if enough time

KANs with splines

Building the Kolmogorov-Arnold model as follows

$$F({}^1x, {}^2x, \cdots, {}^nx) = \sum_{k=1}^{2m+1} \Phi_k \left(\sum_{j=1}^{m} f_{kj}({}^jx) \right)$$

$f_{kj} : [0,1] \mapsto \mathbb{R}$ and $\Phi_k : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions.

The relation is approximated by $k = 1, \cdots, d = 2m+1$ as

$$\hat{F}({}^1x, {}^2x, \cdots, {}^nx) = \sum_{k=1}^{d} \Phi_k \underbrace{\left(\sum_{j=1}^{m} f_{kj}({}^jx_i) \right)}_{\theta_{ik}}$$

where θ_{ik} denotes the k-th component of θ_i vector (interpreted as a hidden variable between two layers), which describes **splines**

M. Poluektov and A. Polar, *"Construction of the Kolmogorov-Arnold representation using the Newton-Kaczmarz method"*, `https://arxiv.org/abs/2305.08194`.

In parting... if enough time

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