Laplace

LABORATOIRE PLASMA ET CONVERSION D'ÉNERGIE

Data-driven multivariate surrogate modelling in the Loewner framework: numerical and practical issues

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Introduction

- System $\rightarrow$ FOM $\rightarrow$ ROM $\rightarrow$ simulation, control

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Reality

Parameter
uncertainties

Model

Introduction

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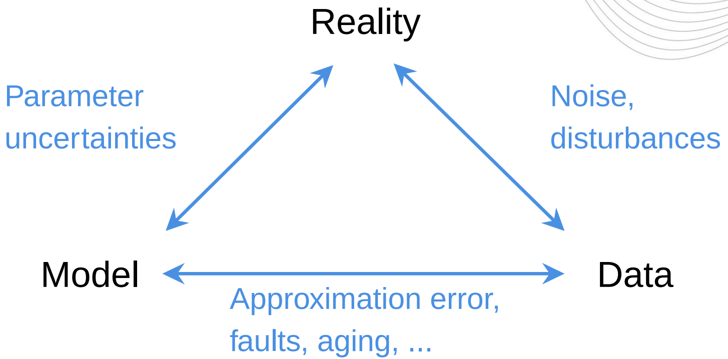

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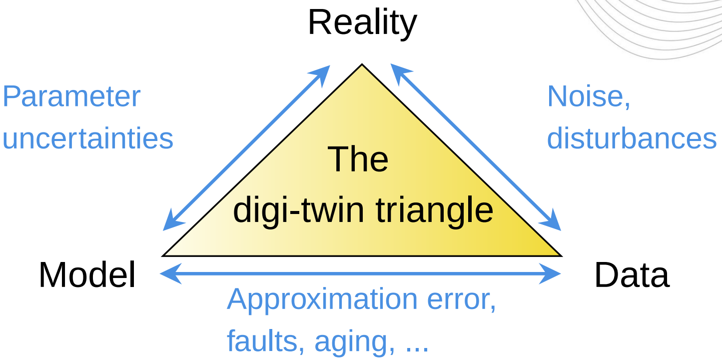

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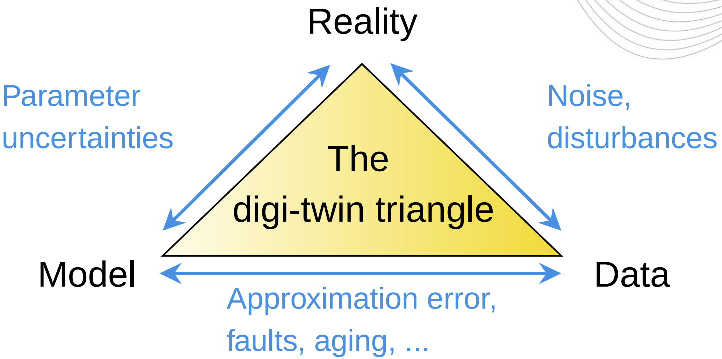

- **Parametric modeling** is very helpful!

Outline

1. **The Loewner framework**
   A. The 1-D case
   B. The 2-D case
   C. The n-D case and the curse of dimensionality
   D. Taming the curse of dimensionality

2. **Numerical challenges**

3. **Conclusions**

The Loewner framework : the 1-D case

Find g such that $\begin{cases} \mathbf{g}({}^1\lambda_{j_1}) & = & \mathbf{w}_{j_1},\ j_1 = 1,\dots,k_1 \\ \mathbf{g}({}^1\mu_{i_1}) & = & \mathbf{v}_{i_1},\ i_1 = 1,\dots,q_1 \end{cases}$

Lagrangian form

$$\mathbf{g}({}^1s) = \frac{\sum_{j_1=1}^{k_1} \frac{c_{j_1}\mathbf{w}_{j_1}}{{}^1s - {}^1\lambda_{j_1}}}{\sum_{j_1=1}^{k_1} \frac{c_{j_1}}{{}^1s - {}^1\lambda_{j_1}}}$$

A tutorial introduction to the Loewner framework for model reduction, Antoulas et al. (2017).

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Loewner matrix

$$\mathbb{L}_1 \in \mathbb{C}^{q_1 \times k_1}$$

$$(\mathbb{L}_1)_{i_1, j_1} = \frac{\mathbf{v}_{i_1} - \mathbf{w}_{j_1}}{{}^1\mu_{i_1} - {}^1\lambda_{j_1}}$$

Null space

$$\mathbf{span}\ (\mathbf{c}_1) = \mathcal{N}(\mathbb{L}_1)$$

$$\mathbf{c}_1 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k_1} \end{bmatrix} \in \mathbb{C}^{k_1}$$

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Extension to the 2-D case

Find g such that $\begin{cases} \mathbf{g}({}^1\lambda_{j_1}, {}^2\lambda_{j_2}) & = & \mathbf{w}_{j_1,j_2}, \; j_1 = 1,\ldots,k_1, \; j_2 = 1,\ldots,k_2 \\ \mathbf{g}({}^1\mu_{i_1}, {}^2\mu_{i_2}) & = & \mathbf{v}_{i_1,i_2}, \; i_1 = 1,\ldots,q_1, \; i_2 = 1,\ldots,q_2 \end{cases}$

Loewner matrix

$$\mathbb{L}_2 \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$$

$$\ell^{i_1,i_2}_{j_1,j_2} = \frac{\mathbf{v}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})({}^2\mu_{i_2} - {}^2\lambda_{j_2})}$$

$$\mathbb{L}_{2D} = \begin{bmatrix} \ell^{1,1\ldots q_2}_{1,1\ldots k_2} & \cdots & \ell^{1,1\ldots q_2}_{k_1,1\ldots k_2} \\ \vdots & \ddots & \vdots \\ \ell^{q_1,1\ldots q_2}_{1,1\ldots k_2} & \cdots & \ell^{q_1,1\ldots q_2}_{k_1,1\ldots k_2} \end{bmatrix}$$

Data-driven parametrized model reduction in the Loewner framework, Ionita and Antoulas (2014).
On two-variable rational interpolation, Antoulas, Ionita, Lefteriu (2012).

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Null space

$$\textbf{span}\ (\mathbf{c}_{2D}) = \mathcal{N}(\mathbb{L}_{2D})$$

$$\mathbf{c}_{2D} = \left[\begin{array}{c} c_{1,1} \\ \vdots \\ c_{1,k_2} \\ \hline \vdots \\ \hline c_{k_1,1} \\ \vdots \\ c_{k_1,k_2} \end{array}\right] \in \mathbb{C}^{k_1 k_2}$$

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$$\ell_{j_1,j_2}^{i_1,i_2} = \frac{\mathbf{v}_{i_1,i_2} - \mathbf{w}_{j_1,j_2}}{({}^1\mu_{i_1} - {}^1\lambda_{j_1})({}^2\mu_{i_2} - {}^2\lambda_{j_2})}$$

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$$\mathbf{g}({}^1s, {}^2s) = \frac{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2} \mathbf{w}_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \frac{c_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}$$

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Driving example (simple 2-D)

Data generated from $\mathbf{H}({}^1s, {}^2s) = \mathbf{H}(s,t)$ of complexity $(1,1)$

$$\mathbf{H}(s,t) = \frac{12\,\gamma - 14\,s - 9\,t - 12\,\gamma\,s - 4\,\gamma\,t + 6\,s\,t + 4\,\gamma\,s\,t + 22}{3\,\gamma - 7\,s - 5\,t - 3\,\gamma\,s - \gamma\,t + 3\,s\,t + \gamma\,s\,t + 12}$$

The parameter γ will be used to highlight numerical issues.

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$$\left.\begin{array}{rcl} {}^1\lambda & = & [1,2] \\ {}^1\mu & = & [6,7] \\ {}^2\lambda & = & [3,2] \\ {}^2\mu & = & [6,7] \end{array}\right\} \xrightarrow{\mathbf{H}} \textbf{tab}_{2D} = \begin{pmatrix} \mathbf{H}({}^1\lambda_1, {}^2\lambda_1) & \mathbf{H}({}^1\lambda_1, {}^2\lambda_2) & \mathbf{H}({}^1\lambda_1, {}^2\mu_1) & \mathbf{H}({}^1\lambda_1, {}^2\mu_2) \\ \mathbf{H}({}^1\lambda_2, {}^2\lambda_1) & \mathbf{H}({}^1\lambda_2, {}^2\lambda_2) & \mathbf{H}({}^1\lambda_2, {}^2\mu_1) & \mathbf{H}({}^1\lambda_2, {}^2\mu_2) \\ \mathbf{H}({}^1\mu_1, {}^2\lambda_1) & \mathbf{H}({}^1\mu_1, {}^2\lambda_2) & \mathbf{H}({}^1\mu_1, {}^2\mu_1) & \mathbf{H}({}^1\mu_1, {}^2\mu_2) \\ \mathbf{H}({}^1\mu_2, {}^2\lambda_1) & \mathbf{H}({}^1\mu_2, {}^2\lambda_2) & \mathbf{H}({}^1\mu_2, {}^2\mu_1) & \mathbf{H}({}^1\mu_2, {}^2\mu_2) \end{pmatrix}$$

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Laplace

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The general n-D case

$$\begin{array}{ccc} \mathbb{C}^{k_1} \times \mathbb{C}^{q_1} \times \ldots \times \mathbb{C}^{k_n} \times \mathbb{C}^{q_n} \times \mathbb{C}^{(k_1+q_1)\times\cdots\times(k_n+q_n)} & \longrightarrow & \mathbb{C}^{Q\times K} \\ \left({}^{1}\lambda_{j_1}, {}^{1}\mu_{i_1}, \ldots, {}^{n}\lambda_{j_n}, {}^{n}\mu_{i_n}, \textbf{tab}_{nD}\right) & \longmapsto & \mathbb{L}_{nD} \end{array}$$

Laplace

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The curse of dimensionality

$$\mathbb{L}_{nD} \in \mathbb{C}^{Q\times K}$$

$Q = q_1 q_2 \ldots q_n$ and $K = k_1 k_2 \ldots k_n$

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Taming the curse of dimensionality

- Iterative solution based on variable decoupling

$$\mathbf{g}({}^1s, {}^2s) = \frac{\sum_{j_1,j_2=1}^{k_1,k_2} \frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}{\sum_{j_1,j_2}^{k_1,k_2} \frac{c_{j_1,j_2}}{({}^1s - {}^1\lambda_{j_1})({}^2s - {}^2\lambda_{j_2})}}$$

$$\mathbf{g}({}^1s, {}^2\lambda_k) = \frac{\mathsf{num}_{{}^2\lambda_k}({}^1s)}{\pi_{{}^2\lambda_k}({}^1s) \sum_{j_1=1}^{k_1} \frac{c_{j_1,k}}{({}^1s - {}^1\lambda_{j_1})}}$$

$$\forall k = 1 \ldots k_2,\ \mathcal{N}(\mathbb{L}_{{}^2\lambda_k}) = \mathsf{span}(c_{1,k} \ \ \ldots \ \ c_{k_1,k})$$

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

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$$\mathbf{g}({}^1s, {}^2\lambda_k) = \frac{\mathsf{num}_{{}^2\lambda_k}({}^1s)}{\pi_{{}^2\lambda_k}({}^1s) \sum_{j_1=1}^{k_1} \frac{c_{j_1,k}}{({}^1s - {}^1\lambda_{j_1})}}$$

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$$\forall k = 1 \dots k_1,\ \mathcal{N}(\mathbb{L}_{{}^1\lambda_k}) = \mathsf{span}(c_{k,1} \ \dots \ c_{k,k_2})$$

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Decoupling process

$$\mathcal{N}(\mathbb{L}_{2D}) = \mathbf{c}_{2D}^{\top} = \left[\begin{array}{ccc|c|ccc} c_{1,1} & \dots & c_{1,k_2} & \dots & c_{k_1,1} & \dots & c_{k_1,k_2} \end{array}\right]^T$$

$$\textbf{tab}_{2D} = \begin{array}{c|c} & \begin{array}{cccc} {}^2s_1 & {}^2s_2 & \dots & {}^2s_m \end{array} \\ \hline \begin{array}{c} {}^1s_1 \\ {}^1s_2 \\ \vdots \\ {}^1s_n \end{array} & \begin{bmatrix} h_{1,1} & h_{1,2} & \dots & h_{1,m} \\ h_{2,1} & h_{2,2} & \dots & h_{2,m} \\ \vdots & \vdots & & \vdots \\ h_{n,1} & h_{n,2} & \dots & h_{n,m} \end{bmatrix} \end{array} \quad \begin{array}{c|c} & \begin{array}{cccc} {}^2\lambda_1 & {}^2\lambda_2 & \dots & {}^2\lambda_{k_2} \end{array} \\ \hline \begin{array}{c} {}^1\lambda_1 \\ {}^1\lambda_2 \\ \vdots \\ {}^1\lambda_{k_1} \end{array} & \begin{bmatrix} c_{1,1} & c_{1,2} & \dots & c_{1,k_2} \\ c_{2,1} & c_{2,2} & \dots & c_{2,k_2} \\ \vdots & \vdots & & \vdots \\ c_{k_1,1} & c_{k_1,2} & \dots & c_{k_1,k_2} \end{bmatrix} \end{array}$$

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$$\textbf{tab}_{2D} = \left[\begin{array}{c|cccc} & {}^2s_1 & {}^2s_2 & \dots & {}^2s_m \\ \hline {}^1s_1 & h_{1,1} & h_{1,2} & \dots & h_{1,m} \\ {}^1s_2 & h_{2,1} & h_{2,2} & \dots & h_{2,m} \\ \vdots & \vdots & \vdots & & \vdots \\ {}^1s_n & h_{n,1} & h_{n,2} & \dots & h_{n,m} \end{array}\right] \quad \left[\begin{array}{c|cccc} & {}^2\lambda_1 & {}^2\lambda_2 & \dots & {}^2\lambda_{k_2} \\ \hline {}^1\lambda_1 & c_{1,1} & c_{1,2} & \dots & c_{1,k_2} \\ {}^1\lambda_2 & c_{2,1} & c_{2,2} & \dots & c_{2,k_2} \\ \vdots & \vdots & \vdots & & \vdots \\ {}^1\lambda_{k_1} & c_{k_1,1} & c_{k_1,2} & \dots & c_{k_1,k_2} \end{array}\right]$$

- **Without decoupling:** 1 $2D$ problem with $\mathbb{L}_{nD} \in \mathbb{C}^{q_1 q_2 \times k_1 k_2}$
- **With decoupling:** k_1 $1D$ problems with $\mathbb{L}_{1D} \in \mathbb{C}^{q_2 \times k_2}$ and 1 with $\mathbb{L}_{1D} \in \mathbb{C}^{q_1 \times k_1}$

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

Decoupling process

- **Step 1:** for $j_1 = 1 \ldots k_1$ (along the first variable)
  - Compute $\mathbf{c}_{1D}^{{}^1\lambda_{j_1}} = \mathcal{N}(\mathbb{L}_{{}^1\lambda_{j_1}})$ for frozen ${}^1s = {}^1\lambda_{j_1}$
  - Scale so that the last element is 1: $\mathbf{c}_{1D}^{{}^1\lambda_{j_1}} = \mathbf{c}_{1D}^{{}^1\lambda_{j_1}} / c_{1D,k_2}^{{}^1\lambda_{j_1}}$

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- **Step 2:** for frozen ${}^2s = {}^2\lambda_{k_2}$ (second variable)
  - Compute the nullspace $\mathbf{c}_{1D}^{{}^2\lambda_{k_2}} = \mathcal{N}(\mathbb{L}_{{}^2\lambda_{k_2}})$
  - Scale it: $\mathbf{c}_{1D}^{{}^2\lambda_{k_2}} = \mathbf{c}_{1D}^{{}^2\lambda_{k_2}} / c_{1D,k_1}^{{}^2\lambda_{k_2}}$

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  - Scale so that the last element is 1: $\mathbf{c}_{1D}^{{}^1\lambda_{j_1}} = \mathbf{c}_{1D}^{{}^1\lambda_{j_1}} / c_{1D,k_2}^{{}^1\lambda_{j_1}}$
- **Step 2:** for frozen ${}^2s = {}^2\lambda_{k_2}$ (second variable)
  - Compute the nullspace $\mathbf{c}_{1D}^{{}^2\lambda_{k_2}} = \mathcal{N}(\mathbb{L}_{{}^2\lambda_{k_2}})$
  - Scale it: $\mathbf{c}_{1D}^{{}^2\lambda_{k_2}} = \mathbf{c}_{1D}^{{}^2\lambda_{k_2}} / c_{1D,k_1}^{{}^2\lambda_{k_2}}$
- **Step 3:** Compute the scaled nullspace

$$\mathbf{c}_{2D}^{\top} = \begin{bmatrix} \mathbf{c}_{1D}^{{}^1\lambda_1} \cdot [\mathbf{c}_{1D,1}^{{}^2\lambda_{k_2}}]_1 & \dots & \mathbf{c}_{1D}^{{}^1\lambda_{k_1}} \cdot [\mathbf{c}_{1D}^{{}^2\lambda_{k_2}}]_{k_2} \end{bmatrix}^{\top}$$

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

Laplace

Decoupling process applied to our driving example

$s \backslash t$	${}^2\lambda_1 = 3$	${}^2\lambda_2 = 2$	${}^2\mu_1 = 6$	${}^2\mu_2 = 7$
${}^1\lambda_1 = 1$	$h_{1,1} = 1$	$h_{1,2} = 2$	$h_{1,3} = \frac{10}{7}$	$h_{1,4} = \frac{13}{9}$
${}^1\lambda_2 = 2$	$h_{2,1} = 3$	$h_{2,2} = 4$	$h_{2,3} = \frac{12\gamma+12}{3\gamma+4}$	$h_{2,4} = \frac{16\gamma+15}{4\gamma+5}$
${}^1\mu_1 = 6$	$h_{3,1} = \frac{19}{9}$	$h_{3,2} = \frac{20\gamma+8}{5\gamma+4}$	$h_{3,3} = \frac{60\gamma+100}{15\gamma+48}$	$h_{3,4} = \frac{80\gamma+127}{20\gamma+61}$
${}^1\mu_2 = 7$	$h_{4,1} = \frac{23}{11}$	$h_{4,2} = \frac{24\gamma+10}{6\gamma+5}$	$h_{4,3} = \frac{72\gamma+122}{18\gamma+59}$	$h_{4,4} = \frac{96\gamma+155}{24\gamma+75}$

$$\xrightarrow{\mathcal{N}(\mathbb{L}_{2D})} \mathbf{c}_2 = \begin{pmatrix} \frac{1}{\gamma} \\ \frac{1}{\gamma} \\ \frac{1}{\gamma} \\ 1 \end{pmatrix}$$

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

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Laplace

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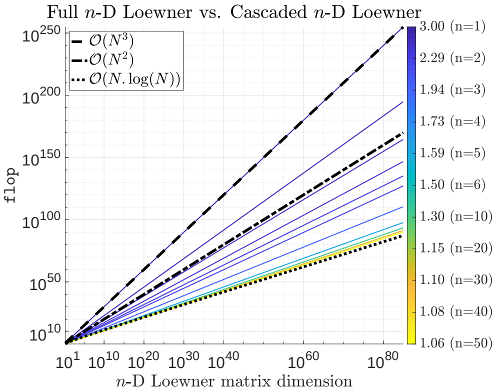
$$\text{Scaled null space } \mathbf{c}_2^\top = \begin{bmatrix} \mathbf{c}_1^{{}^1\lambda_1} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_1 & \mathbf{c}_1^{{}^1\lambda_2} \cdot [\mathbf{c}_1^{{}^2\lambda_2}]_2 \end{bmatrix}^\top$$

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

Taming the curse of dimensionality

n-D `flop` and MB

log-log scale



Computational issue

$Q \times K$ matrix SVD `flop` is

- QK^2 (if $Q > K$)
- N^3 (if $Q = K = N$)

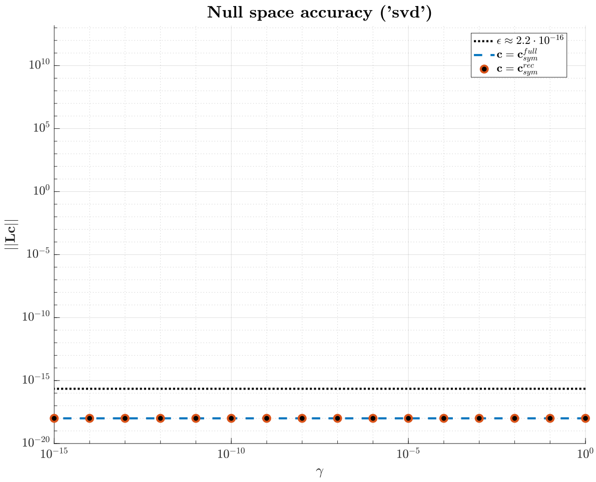
Storage issue

$Q \times K$ matrix storage is

- in real double
- in complex double

Laplace

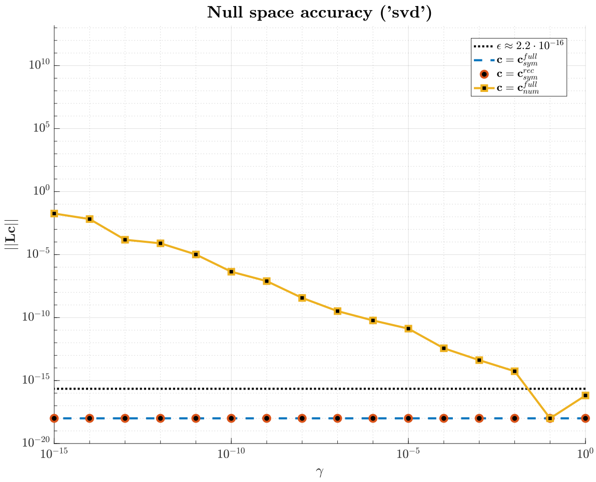
Numerical challenges



Impact of γ on $||\mathbb{L}\mathbf{c}|| \stackrel{?}{=} 0$

- In exact arithmetic, no approximation error no matter γ

Numerical challenges

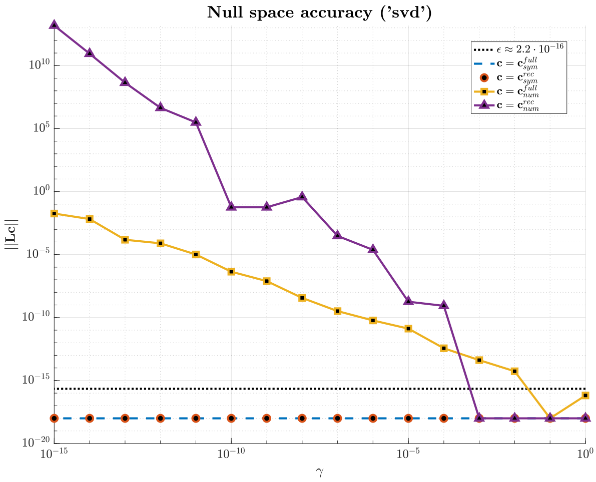


Impact of γ on $||\mathbb{L}\mathbf{c}|| \stackrel{?}{=} 0$

- In exact arithmetic, no approximation error no matter γ
- When solving numerically, the accuracy $\searrow$ when $\gamma \searrow$ for the full 2D method

Laplace

Numerical challenges



Impact of γ on $||\mathbb{L}\mathbf{c}|| \overset{?}{=} 0$

- In exact arithmetic, no approximation error no matter γ
- When solving numerically, the accuracy $\searrow$ when $\gamma \searrow$ for the full 2D method
- It's far worse for the recursive 1D procedure!

Laplace

Numerical challenges

Change of scaling element

Normalization wrt. the last element

$$\mathbf{c}_1^{2\lambda_2} = \begin{bmatrix} \frac{1}{\gamma} \\ 1 \end{bmatrix}, \mathbf{c}_1^{1\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{c}_1^{1\lambda_2} = \begin{bmatrix} \frac{1}{\gamma} \\ 1 \end{bmatrix} \mathcal{N}(\mathbb{L}_{2D}) = \begin{pmatrix} \frac{1}{\gamma} \\ \frac{1}{\gamma} \\ \frac{1}{\gamma} \\ 1 \end{pmatrix}$$

Numerical challenges

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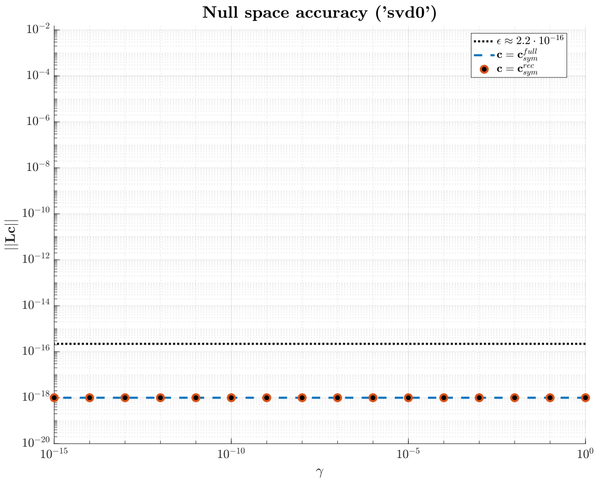
Normalization wrt. the highest element

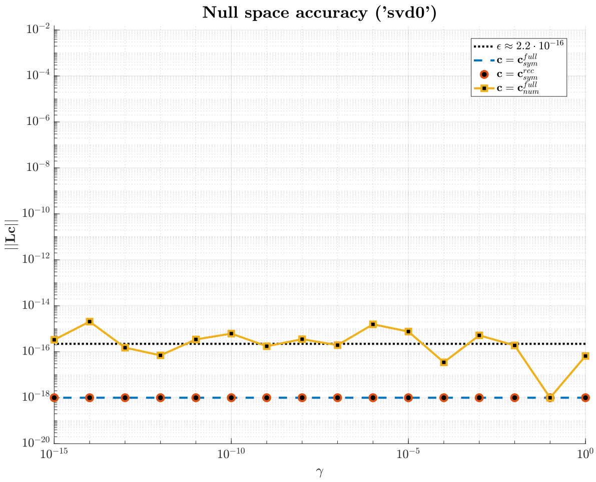
$$\mathbf{c}_1^{2\lambda_2} = \begin{bmatrix} 1 \\ \gamma \end{bmatrix}, \mathbf{c}_1^{1\lambda_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{c}_1^{1\lambda_2} = \begin{bmatrix} \frac{1}{\gamma} \\ 1 \end{bmatrix} \mathcal{N}(\mathbb{L}_{2D}) = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \gamma \end{pmatrix}$$

Numerical challenges



Laplace

Numerical challenges

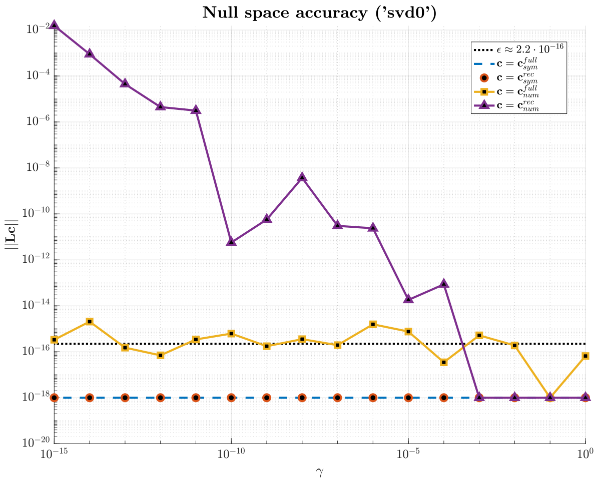


- Improvement for the full 2D method

$$10^{-2} \rightarrow 10^{-16}$$

Laplace

Numerical challenges



- Improvement for the full 2D method

$$10^{-2} \rightarrow 10^{-16}$$

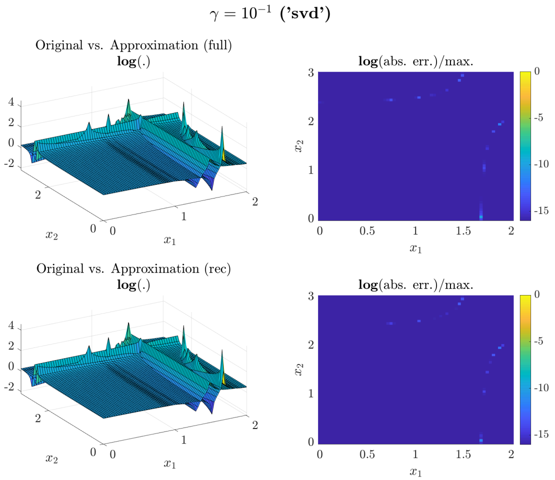
- and for the recursive 1D method

$$10^{13} \rightarrow 10^{-2}$$

Laplace

Numerical challenges

Evaluation



Laplace

Conclusions

- Efficient recursive method for multivariate rational interpolation
- Requires more data than the $n - D$ procedure

$$\underbrace{\begin{pmatrix} 1 & 2 & \frac{10}{7} & \frac{13}{9} \\ 3 & 4 & \frac{12\gamma+12}{3\gamma+4} & \frac{16\gamma+15}{4\gamma+5} \\ \frac{19}{9} & \frac{20\gamma+8}{5\gamma+4} & \frac{60\gamma+100}{15\gamma+48} & \frac{80\gamma+127}{20\gamma+61} \\ \frac{23}{11} & \frac{24\gamma+10}{6\gamma+5} & \frac{72\gamma+122}{18\gamma+59} & \frac{96\gamma+155}{24\gamma+75} \end{pmatrix}}^{\textbf{tab}_{2D}} \quad \underbrace{\begin{pmatrix} 1 & 2 & \frac{10}{7} & \frac{13}{9} \\ 3 & 4 & \frac{12\gamma+12}{3\gamma+4} & \frac{16\gamma+15}{4\gamma+5} \\ \frac{19}{9} & \frac{20\gamma+8}{5\gamma+4} & \frac{60\gamma+100}{15\gamma+48} & \frac{80\gamma+127}{20\gamma+61} \\ \frac{23}{11} & \frac{24\gamma+10}{6\gamma+5} & \frac{72\gamma+122}{18\gamma+59} & \frac{96\gamma+155}{24\gamma+75} \end{pmatrix}}^{\textbf{tab}_{rec-1D}}$$

- Compared to learning, “incomplete” data is not an option
- The recursive algorithm introduces some numerical errors that can be mitigated by scaling the nullspace according to the highest element.

Outlooks

- More work is needed to meet the control formalism.

Mono-variable case

$$\begin{cases} \mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y = \mathbf{C}\mathbf{x} \end{cases}$$

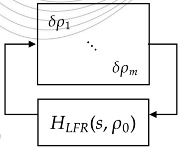
$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$$

LPV case

$$\begin{cases} \mathbf{E}(\rho)\dot{\mathbf{x}} = \mathbf{A}(\rho)\mathbf{x} + \mathbf{B}(\rho)u \\ y = \mathbf{C}(\rho)\mathbf{x} \end{cases}$$

$$\mathbf{H}(s,\rho) = \mathbf{C}(\rho)(s\mathbf{E}(\rho) - \mathbf{A}(\rho))^{-1}\mathbf{B}(\rho)$$

LFR representation



A data-driven, noise-resilient algorithm for extraction of distribution of relaxation times using the Loewner framework, Patel et al., 2025

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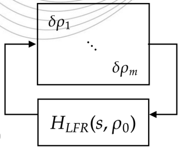
LFR representation



- Hard to use on noisy data (as in the monovariable case)
  - Noise makes it hard to select a reduction order
  - Play with data ordering
  - Classify the SV according to the corresponding residues
  - ScreeNOT + AIC filtering

A data-driven, noise-resilient algorithm for extraction of distribution of relaxation times using the Loewner framework, Patel et al., 2025

Connection to the Kolmogorov Superposition Theorem

Kolmogorov, Arnol'd, Kahane, Lorentz, and Sprecher

For any $n \in \mathbb{N}$, $n \geq 2$, there exist real numbers $\lambda_1, \ldots, \lambda_n$ and continuous functions $\Phi_k : \mathbb{I} \mapsto \mathbb{R}$, $k = 1, \ldots, 2n+1$, with the property that for every continuous function $f : \mathbb{I}^n \mapsto \mathbb{R}$, there exists a continuous function $g : \mathbb{R} \mapsto \mathbb{R}$ such that:

$$\forall (x_1, \ldots, x_n) \in \mathbb{I}^n, \quad f(x_1, \ldots, x_n) = \sum_{k=1}^{2n+1} g(\lambda_1 \Phi_k(x_1) + \cdots + \lambda_n \Phi_k(x_n))$$

Hilbert 13: Are there any genuine continuous multivariate real-valued functions?, S. Morris, (2021).

KST-like representation for rational functions

- Thanks to the decoupling process, one can write :

$$\mathbf{H}(s,t)=\frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\frac{c_{j_1,j_2}\mathbf{w}_{j_1,j_2}}{(s-{}^1\lambda_{j_1})(t-{}^2\lambda_{j_2})}}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\frac{c_{j_1,j_2}}{(s-{}^1\lambda_{j_1})(t-{}^2\lambda_{j_2})}}$$

$$=\frac{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\exp\left(\log\mathbf{w}_{j_1,j_2}+\log\frac{\mathbf{Bary}_{j_1}^{s}}{(s-{}^1\lambda_{j_1})}+\log\frac{\mathbf{Bary}_{j_2}^{t}}{(t-{}^2\lambda_{j_2})}\right)}{\sum_{j_1=1}^{k_1}\sum_{j_2=1}^{k_2}\exp\left(\log\frac{\mathbf{Bary}_{j_1}^{s}}{(s-{}^1\lambda_{j_1})}+\log\frac{\mathbf{Bary}_{j_2}^{t}}{(t-{}^2\lambda_{j_2})}\right)}$$

- Composition and superposition of one-variable functions

On the Loewner framework, the Kolmogorov superposition theorem, and the curse of dimensionality, Antoulas, Gosea, Poussot-Vassal (2024).

Decoupling and equivalent KST

- Remember that $\mathbf{c}_{2D}^{\top} = \left[\begin{array}{ccc} {\mathbf{c}_{1D}}^{1\lambda_1} \cdot [{\mathbf{c}_{1D,1}}^{2\lambda_{k_2}}]_1 & \dots & {\mathbf{c}_{1D}}^{1\lambda_{k_1}} \cdot [{\mathbf{c}_{1D}}^{2\lambda_{k_2}}]_{k_2} \end{array}\right]^{\top}$ (decoupling)

Decoupling and equivalent KST

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- Barycentric weights can be rewritten as

$$\mathbf{c}^s = \begin{pmatrix} \mathbf{c}_{1D}^{{}^1\lambda_1} \\ \\ \mathbf{c}_{1D}^{{}^1\lambda_{k_1}} \end{pmatrix} \quad \text{and } \mathbf{Bary}_s = \mathbf{c}^t$$

$$\mathbf{c}^t = \mathbf{c}_{1D}^{{}^2\lambda_{k_2}} \quad \text{and } \mathbf{Bary}_t = \mathbf{c}^t \otimes \mathbf{1}_{k_2}$$

Laplace

Decoupling and equivalent KST

- Remember that $\mathbf{c}_{2D}^{\top} = \begin{bmatrix} \mathbf{c}_{1D}^{{}^1\lambda_1} \cdot [\mathbf{c}_{1D,1}^{{}^2\lambda_{k_2}}]_1 & \dots & \mathbf{c}_{1D}^{{}^1\lambda_{k_1}} \cdot [\mathbf{c}_{1D}^{{}^2\lambda_{k_2}}]_{k_2} \end{bmatrix}^{\top}$ (decoupling)
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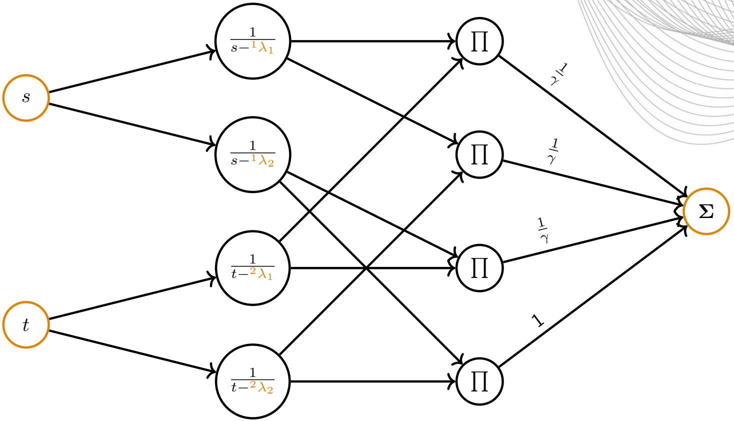
$$\mathbf{c}^t = \mathbf{c}_{1D}^{{}^2\lambda_{k_2}} \qquad \text{ and } \mathbf{Bary}_t = \mathbf{c}^t \otimes \mathbf{1}_{k_2}$$

$$\mathbf{c} = \mathbf{Bary}_t \odot \mathbf{Bary}_s$$

This is (s,t) variables decoupling!

Connection with neural networks



NN with barycentric activation functions and product/sum

Kan: Kolmogorov-arnold networks, Liu et al. (2024).

Laplace